

# $G$ -GERBES, PRINCIPAL 2-GROUP BUNDLES AND CHARACTERISTIC CLASSES

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## Abstract

We give an explicit description of a 1-1 correspondence between Morita equivalence classes of, on the one hand, principal 2-group  $[G \rightarrow \text{Aut}(G)]$ -bundles over Lie groupoids and, on the other hand,  $G$ -extensions of Lie groupoids (i.e. between  $[G \rightarrow \text{Aut}(G)]$ -bundles over differentiable stacks and  $G$ -gerbes over differentiable stacks). We also introduce universal characteristic classes for 2-group bundles. For groupoid central  $G$ -extensions, we prove that the universal characteristic classes coincide with the Dixmier Douady classes that can be computed from connection-type data.

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# 1 Introduction

This paper is devoted to the study of the relation between groupoid  $G$ -extensions and *principal Lie 2-group bundles*, and of their characteristic classes.

A *Lie 2-group* is a Lie groupoid  $\Gamma_2 \rightrightarrows \Gamma_1$ , whose spaces of objects  $\Gamma_1$  and of morphisms  $\Gamma_2$  are Lie groups and all of whose structure maps are group morphisms. A *crossed module*  $(G \xrightarrow{\rho} H)$  is a Lie group morphism  $G \xrightarrow{\rho} H$  together with an action of  $H$  on  $G$  satisfying suitable compatibility conditions. It is standard that Lie 2-groups are in bijection with crossed modules [3, 12]. In this paper,  $[G \xrightarrow{\rho} H]$  denotes the 2-group corresponding to the crossed module  $(G \xrightarrow{\rho} H)$ .

Lie 2-groups arise naturally in mathematical physics. For instance, in higher gauge theory [2, 4], Lie 2-group bundles provide a well suited framework for describing the parallel transport of strings [1, 4, 29]. Several recent works have approached the concept of bundles with a “structure Lie 2-group” over a manifold from various perspectives [1, 4, 6, 36]. Here we take an alternative point of view and give a definition of principal Lie 2-group bundles of a global nature (i.e. not resorting to a description explicitly involving local charts and cocycles) and which allows for the base space to be a Lie groupoid. In other words, we consider 2-group principal bundles over differentiable stacks [7]. Our approach immediately leads to a natural construction of “universal characteristic classes” for principal 2-group bundles.

Let us start with Lie (1-)groups. A principal  $G$ -bundle  $P$  over a manifold  $M$  canonically determines a homotopy class of maps from  $M$  to the classifying space  $BG$  of the group  $G$ . In fact, the set of isomorphism classes of  $G$ -principal bundles over  $M$  is in bijection with the set of homotopy classes of maps  $M \xrightarrow{f} BG$  [14, 38, 39]. Pulling back the generators of  $H^*(BG)$  (the universal classes) through  $f$ , one obtains characteristic classes of the principal bundle  $P$  over  $M$ . These characteristic classes coincide with those obtained from a connection by applying the Chern-Weil construction [17, 32].

There is an analogue but much less known, differential geometric rather than purely topological, point of view: a principal  $G$ -bundle over a manifold  $M$  can be thought of as a “generalized morphism” (in the sense of Hilsum-Skandalis [24]) from the manifold  $M$  to the Lie group  $G$  both considered as 1-groupoids. To see this, recall that a principal  $G$ -bundle can be defined as a collection of transition functions  $g_{ij} : U_{ij} \rightarrow G$  on the double intersections of some open covering, satisfying the cocycle condition  $g_{ij}g_{jk} = g_{ik}$ . These transition functions constitute a morphism of groupoids from the Čech groupoid  $\coprod U_{ij} \rightrightarrows \coprod U_i$  associated to the open covering  $\{U_i\}_{i \in I}$  to the Lie group  $G \rightrightarrows *$ . Hence we have a diagram

$$(M \rightrightarrows M) \xleftarrow{\sim} (\coprod U_{ij} \rightrightarrows \coprod U_i) \rightarrow (G \rightrightarrows *)$$

in the category of Lie groupoids and their morphisms whose leftward arrow is a Morita equivalence, in other words a generalized morphism from the manifold  $M$  to the Lie group  $G$ .

This second point of view, or more precisely its generalization to the 2-groupoid context, constitutes the foundations on which our approach is built. The generalization of the concept of “generalized morphism” to 2-groupoids is straightforward: a generalized morphism of Lie 2-groupoids  $\Gamma \rightsquigarrow \Delta$  is a diagram  $\Gamma \xleftarrow[\sim]{\phi} \mathbf{E} \xrightarrow{f} \Delta$  in the category **2Gpd** of Lie 2-groupoids and their morphisms, where  $\phi$  is a

Morita equivalence (a “smooth” equivalence of 2-groupoids). It is sometimes useful to think of two Morita equivalent Lie 2-groupoids as two different choices of an atlas (or open cover) on the same geometric object (which is a differentiable 2-stack [7, 11]). We define a principal  $[G \xrightarrow{\rho} H]$ -bundle over a Lie groupoid  $\Gamma$  to be a generalized morphism from  $\Gamma$  to  $[G \xrightarrow{\rho} H]$  (up to equivalence). See Section 2.3.

The concept of (geometric) nerve of Lie groupoids extends to the 2-categorical context as a functor from the category of Lie 2-groupoids to the category of simplicial manifolds [40]. By convention, the cohomology of a 2-groupoid is the cohomology of its nerve, which can be computed via a double complex (for instance, see [20]). Crucially, Morita equivalences induce isomorphisms in cohomology. Therefore, any generalized morphism of 2-groupoids  $\Gamma \xrightarrow{F} [G \rightarrow H]$  defining a principal  $[G \rightarrow H]$ -bundle  $\mathfrak{B}$  over the groupoid  $\Gamma$  yields a pullback homomorphism  $F^* : H^\bullet([G \rightarrow H]) \rightarrow H^\bullet(\Gamma)$  in cohomology, which is called the *cohomology characteristic map* (characteristic map for short). The cohomology classes in  $H^\bullet([G \rightarrow H])$  [20] should be viewed as universal characteristic classes and their images by  $F^*$  as the characteristic classes of  $\mathfrak{B}$ .

Lie 2-group principal bundles are closely related to non-abelian gerbes. Geometrically, non-abelian  $G$ -gerbes over differentiable stacks can be considered as groupoid  $G$ -extensions modulo Morita equivalence [26]. By a groupoid  $G$ -extension, we mean a short exact sequence of groupoids  $1 \rightarrow M \times G \xrightarrow{i} \tilde{\Gamma} \xrightarrow{\phi} \Gamma \rightarrow 1$ , where  $M \times G$  is a bundle of groups. Here we establish an explicit 1-1 correspondence between groupoid  $G$ -extensions up to Morita equivalence (i.e.  $G$ -gerbes over differentiable stacks) and principal  $[G \rightarrow \text{Aut}(G)]$ -bundles over Lie groupoids modulo Morita equivalence (i.e.  $[G \rightarrow \text{Aut}(G)]$ -principal bundles over differentiable stacks). This is Theorem 3.4. Note that a restricted version of this correspondence is highlighted in [27, Theorem 4].

It is known that Giraud’s second non abelian cohomology group  $H^2(\mathfrak{X}, G)$  classifies the  $G$ -gerbes over a differentiable stack  $\mathfrak{X}$  [21] while Dedeckers’  $H^1(\mathfrak{X}, [G \rightarrow \text{Aut}(G)])$  classifies the principal  $[G \rightarrow \text{Aut}(G)]$ -bundles [15, 16]. In [9, 10], Breen showed that these two cohomology groups are isomorphic. In some sense, our theorem above can be considered as an explicit geometric proof of Breen’s theorem in the smooth context. Indeed, one of the main motivations behind the present paper is the relation between  $G$ -extensions and 2-group principal bundles. We believe our result throws a bridge between the groupoid extension approach to the differential geometry of  $G$ -gerbes developed in [26] and the one based on higher gauge theory due to Baez-Schreiber [4]. This will be investigated somewhere else.

An important class of  $G$ -extensions is formed by the so called *central  $G$ -extensions* [26], those for which the structure 2-group  $[G \rightarrow \text{Aut}(G)]$  reduces to the 2-group  $[Z(G) \rightarrow 1]$  (where  $Z(G)$  stands for the center of  $G$ ). They correspond to  $G$ -gerbes with trivial band or  $G$ -bound gerbes [26]. Each such extension determines a principal  $[Z(G) \rightarrow 1]$ -bundle over the base groupoid  $\Gamma$ . In [7], Behrend-Xu gave a natural construction associating a class in  $H^3(\Gamma)$  to a central  $S^1$ -extension of a Lie groupoid  $\Gamma$ . When the base Lie groupoid is Morita equivalent to a smooth manifold (viewed as a trivial 2-groupoid), a central  $S^1$ -extension is what has been studied by Murray and Hitchin under the name bundle gerbe [25, 35]. The Behrend-Xu class of a bundle gerbe coincides with its *Dixmier-Douady class*, which can be described by the 3-curvature. In the present paper, we extend the construction of Behrend-Xu and define a Dixmier-Douady class  $\mathbf{DD}_{(\alpha)} \in H^3(\Gamma) \otimes Z(\mathfrak{g})$  for any central  $G$ -extension, where  $G$  is a connected reductive Lie group. Since a central  $G$ -extension induces a  $[Z(G) \rightarrow 1]$ -principal bundle over

$\Gamma$ , there is also a characteristic map  $H^3([Z(G) \rightarrow 1]) \rightarrow H^3(\Gamma)$ . Dualizing, one obtains a class  $\mathbf{CC}_\phi \in H^3(\Gamma) \otimes Z(\mathfrak{g})$ . We prove that the Dixmier-Douady class  $\mathbf{DD}_{(\alpha)}$  coincides with the characteristic class  $\mathbf{CC}_\phi$ . In a certain sense, this is the gerbe analogue of the Chern-Weil isomorphism for principal bundles [17, 32].

The paper is organized as follows. Section 2 is concerned with generalized morphisms of Lie 2-groupoids and with 2-group bundles and recalls some standard material on Lie 2-groupoids. The main feature of Section 3 is Theorem 3.4 on the equivalence of groupoids  $G$ -extensions and principal  $[G \xrightarrow{\text{Ad}} \text{Aut}(G)]$ -bundles. In Section 4 we define the characteristic map/classes of principal Lie 2-group bundles, we present the construction of the Dixmier-Douady classes of groupoid central  $G$ -extensions and we prove that the Dixmier-Douady class of a central  $G$ -extension coincides with the universal characteristic class of the induced  $[Z(G) \rightarrow 1]$ -bundle — see Theorem 4.14.

Note that, when  $G$  is discrete, the relation between groupoid  $G$ -extensions and 2-group principal bundles was also independently studied by Haefliger [22].

Some of the results of the present paper are related to results announced by Baez Stevenson [5]. Recently, Sati, Stasheff and Schreiber have studied characteristic classes for 2-group bundles by the mean of  $L_\infty$ -algebras [36]. It would be very interesting to relate their construction to ours using integration of  $L_\infty$ -algebras as in [19, 23].

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## 2 Generalized morphisms and principal Lie 2-group bundles

### 2.1 Lie 2-groupoids, Crossed modules and Morita morphisms

This section is concerned with Lie 2-groupoids and Morita equivalences. The material is rather standard. For instance, see [31, 34] for the general theory of Lie groupoids and [3, 42] for Lie 2-groupoids. A *Lie 2-groupoid* is a double Lie groupoid

$$\begin{array}{ccc} \Gamma_2 & \begin{array}{c} \xrightarrow{s} \\ \rightrightarrows \\ \xleftarrow{t} \end{array} & \Gamma_0 \\ \begin{array}{c} u \downarrow \\ \parallel \\ l \downarrow \end{array} & & \begin{array}{c} \text{id} \downarrow \\ \parallel \\ \text{id} \downarrow \end{array} \\ \Gamma_1 & \begin{array}{c} \xrightarrow{s} \\ \rightrightarrows \\ \xleftarrow{t} \end{array} & \Gamma_0 \end{array} \quad (1)$$

in the sense of [12], where the right column  $\Gamma_0 \begin{array}{c} \xrightarrow{\text{id}} \\ \rightrightarrows \\ \xleftarrow{\text{id}} \end{array} \Gamma_0$  denotes the trivial groupoid associated to the smooth manifold  $\Gamma_0$ . It makes sense to use the symbols  $s$  and  $t$  to denote the source and target maps of the groupoid  $\Gamma_2 \rightrightarrows \Gamma_0$  since  $s \circ l = s \circ u$  and  $t \circ l = t \circ u$ .

**Remark 2.1.** A *Lie 2-groupoid* is thus a small 2-category in which all arrows are invertible, the sets of objects, 1-arrows and 2-arrows are smooth manifolds, all structure maps are smooth and the sources and targets are surjective submersions.

In the sequel, the 2-groupoid (1) will be denoted  $\Gamma_2 \xrightarrow[u]{l} \Gamma_1 \xrightarrow[t]{s} \Gamma_0$ . The so called vertical (resp. horizontal) multiplication in the groupoid  $\Gamma_2 \xrightarrow[u]{l} \Gamma_1$  (resp.  $\Gamma_2 \xrightarrow[t]{s} \Gamma_0$ ) will be denoted by  $\star$  (resp.  $*$ )

Clearly, a Lie groupoid can be seen as a Lie 2-groupoid  $\Gamma_1 \xrightarrow[\text{id}]{\text{id}} \Gamma_1 \xrightarrow[t]{s} \Gamma_0$ .

A Lie 2-groupoid where  $\Gamma_0$  is the one-point space  $*$  is known as a *Lie 2-group*.

There is a well-known equivalence between Lie 2-groupoids and crossed modules of groupoids [12]. A *crossed module* of groupoids is a morphism of groupoids

$$\begin{array}{ccc} X_1 & \xrightarrow{\rho} & \Gamma_1 \\ \Downarrow & & \Downarrow \\ X_0 & \xrightarrow{=} & \Gamma_0 \end{array}$$

which is the identity on the base spaces (i.e.  $X_0 = \Gamma_0$ ) and where  $X_1 \rightrightarrows X_0$  is a family of groups (i.e. source and target maps coincide), together with a right action by automorphisms  $(\gamma, x) \mapsto x^\gamma$  of  $\Gamma$  on  $X$  satisfying:

$$\rho(x^\gamma) = \gamma^{-1} \rho(x) \gamma \quad \forall (x, \gamma) \in X_1 \times_{\Gamma_0} \Gamma_1, \quad (2)$$

$$x^{\rho(y)} = y^{-1} x y \quad \forall (x, y) \in X_1 \times_{\Gamma_0} X_1. \quad (3)$$

Note that the equalities (2) and (3) make sense because  $X_1$  is a family of groups.

**Example 2.2.** Given any Lie group  $G$ , we obtain a crossed module by setting  $N_1 = G$ ,  $\Gamma_1 = \text{Aut}(G)$ ,  $\Gamma_0 = *$  and  $\rho(g) = \mathbf{Ad}_g$  (the conjugation by  $g$ ).

**Example 2.3.** A Lie groupoid  $\Gamma_1 \rightrightarrows \Gamma_0$  induces a crossed module in the following way. Let  $S_\Gamma = \{x \in \Gamma_1 / s(x) = t(x)\}$  be the set of closed loops in  $\Gamma_1$ . Then  $S_\Gamma$  is a family of groups over  $\Gamma_0$  and  $\Gamma_1$  acts by conjugation on  $S_\Gamma$ . Therefore, we obtain a crossed module

$$\begin{array}{ccc} S_\Gamma & \xrightarrow{i} & \Gamma_1 \\ \Downarrow & & \Downarrow \\ \Gamma_0 & \xrightarrow{=} & \Gamma_0 \end{array}$$

where  $i$  is the inclusion map.

A 2-groupoid  $\Gamma_2 \xrightarrow[u]{l} \Gamma_1 \xrightarrow[t]{s} \Gamma_0$  determines a crossed module of groupoids

$(G \xrightarrow{\rho} H)$  as follows. Here  $H = \Gamma_1 \rightrightarrows \Gamma_0$ ,  $G_1 = \{g \in \Gamma_2 / l(g) \in \Gamma_0 \subset \Gamma_1\}$ ,  $\rho(g) = u(g)$  and the action of  $H_1 = \Gamma_1$  on  $G_1 \subset \Gamma_2$  is by conjugation. More precisely, if  $1_h$  is the unit over an object  $h$  in the groupoid  $\Gamma_2 \xrightarrow[u]{l} \Gamma_1$ , then  $g^h = 1_{h^{-1}} * g * 1_h$ .

Conversely, given a crossed module of groupoids  $X \xrightarrow{\rho} \Gamma$ , one gets a Lie 2-groupoid  $X_1 \times \Gamma_1 \xrightarrow[u]{l} \Gamma_1 \xrightarrow[t]{s} \Gamma_0$ , where  $X_1 \times \Gamma_1 \rightrightarrows \Gamma_1$  is the transformation groupoid and  $X_1 \times \Gamma_1 \rightrightarrows \Gamma_0$  is the semi-direct product of groupoids. More precisely, for all  $x, x' \in X_1$  and  $\gamma, \gamma' \in \Gamma_1$ , the structures maps are defined by

$$\begin{aligned} l(x, \gamma) &= \gamma, & (x', \gamma') * (x, \gamma) &= (x' x^{\gamma'^{-1}}, \gamma' \gamma), \\ u(x, \gamma) &= \rho(x) \gamma, & (x', \rho(x) \gamma) \star (x, \gamma) &= (x' x, \gamma). \end{aligned}$$

In the sequel, we will denote the Lie 2-groupoid associated to the crossed module  $(G \xrightarrow{\rho} H)$  by  $[G \xrightarrow{\rho} H]$ .

**Example 2.4.** The crossed module of groups  $(G \xrightarrow{\mathbf{Ad}} \text{Aut}(G))$  yields the 2-group  $G \ltimes \text{Aut}(G) \xrightarrow[l]{u} \text{Aut}(G) \rightrightarrows *$  with structure maps

$$\begin{aligned} l(g, \varphi) &= \varphi & u(g, \phi) &= \mathbf{Ad}_g \circ \varphi \\ (g_1, \mathbf{Ad}_{g_2} \circ \varphi_2) \star (g_2, \varphi_2) &= (g_1 g_2, \varphi_2) \\ (g_1, \varphi_1) * (g_2, \varphi_2) &= (g_1 \varphi_1(g_2), \varphi_1 \circ \varphi_2) \end{aligned}$$

A (strict) morphism  $\mathbf{\Gamma} \xrightarrow{\phi} \mathbf{\Delta}$  of Lie 2-groupoids is a triple  $(\phi_0, \phi_1, \phi_2)$  of smooth maps  $\phi_i : \Gamma_i \rightarrow \Delta_i$  ( $i = 0, 1, 2$ ) commuting with all structure maps. Morphisms of crossed modules are defined similarly.

Let  $\mathbf{\Delta}$  be a Lie 2-groupoid. Given a surjective submersion  $f : M \rightarrow \Delta_0$ , we can form the pullback Lie 2-groupoid  $\mathbf{\Delta}[M] : \Delta_2[M] \xrightarrow[l]{u} \Delta_1[M] \xrightarrow[t]{s} M$ , where

$$\Delta_i[M] = \{(m, \gamma, n) \in M \times \Delta_i \times M \text{ s.t. } s(\gamma) = f(m), t(\gamma) = f(n)\}, \quad i = 1, 2.$$

The maps  $s, t$  are the projections on the first and last factor respectively. The maps  $u, l$ , the horizontal and vertical multiplications are induced by the ones on  $\mathbf{\Delta}$  as follows:

$$\begin{aligned} u(m, \gamma, n) &= (m, u(\gamma), n), & (m, \gamma, n) * (n, \gamma', p) &= (m, \gamma * \gamma', p), \\ l(m, \gamma, n) &= (m, l(\gamma), n), & (m, \gamma, n) \star (m, \gamma', n) &= (m, \gamma \star \gamma', n). \end{aligned}$$

There is a natural map of groupoids  $\mathbf{\Delta}[M] \rightarrow \mathbf{\Delta}$  defined by  $m \mapsto f(m)$  and  $(m, \gamma, n) \mapsto \gamma$ .

Pullback of 2-groupoids yield a convenient definition of Morita morphism of Lie 2-groupoids.

**Definition 2.5.** A morphism of Lie 2-groupoids  $\mathbf{\Gamma} \xrightarrow{\phi} \mathbf{\Delta}$  is a Morita morphism if  $\phi$  is the composition of two morphisms

$$\begin{array}{ccccc} \Gamma_2 & \longrightarrow & \Delta_2[\Gamma_0] & \longrightarrow & \Delta_2 \\ \Downarrow & & \Downarrow & & \Downarrow \\ \Gamma_1 & \longrightarrow & \Delta_1[\Gamma_0] & \longrightarrow & \Delta_1 \\ \Downarrow & & \Downarrow & & \Downarrow \\ \Gamma_0 & \xrightarrow{\text{id}} & \Gamma_0 & \xrightarrow{\phi_0} & \Delta_0 \end{array}$$

such that  $\Gamma_0 \rightarrow \Delta_0$  and  $\Gamma_1 \rightarrow \Delta_1[\Gamma_0]$  are surjective submersions and

$$\begin{array}{ccc} \Gamma_2 & \longrightarrow & \Delta_2[\Gamma_0] \\ \Downarrow & & \Downarrow \\ \Gamma_1 & \longrightarrow & \Delta_1[\Gamma_0] \end{array}$$

is a Morita morphism of 1-groupoids.

The (weakest) equivalence relation generated by the Morita morphism is called *Morita equivalence*. More precisely, two Lie 2-groupoids  $\mathbf{\Gamma}$  and  $\mathbf{\Delta}$  are Morita equivalent if there exists a finite collection  $\mathbf{E}_0, \mathbf{E}_1, \dots, \mathbf{E}_n$  of Lie 2-groupoids with  $\mathbf{E}_0 = \mathbf{\Gamma}$  and  $\mathbf{E}_n = \mathbf{\Delta}$ , and, for each  $i \in \{1, \dots, n\}$ , either a Morita morphism  $\mathbf{E}_{i-1} \rightarrow \mathbf{E}_i$  or a Morita morphism  $\mathbf{E}_i \rightarrow \mathbf{E}_{i-1}$ . In fact, by Lemma 2.12(d), one has the following well-known lemma.

**Lemma 2.6.** *If  $\mathbf{\Gamma}$  and  $\mathbf{\Delta}$  are Morita equivalent, there exists a chain of Morita morphisms  $\mathbf{\Gamma} \leftarrow \mathbf{E} \rightarrow \mathbf{\Delta}$  of length 2 in between  $\mathbf{\Gamma}$  and  $\mathbf{\Delta}$ .*

**Remark 2.7.** *In the categorical point of view, a Morita morphism  $\phi : \mathbf{\Gamma} \rightarrow \mathbf{\Delta}$  is in particular a 2-equivalence of 2-categories preserving the smooth structures.*

**Remark 2.8.** *Similar to [7], one can define differentiable 2-stacks. Two Lie 2-groupoids define the same differentiable 2-stack if, and only if, they are Morita equivalent. In fact a Lie 2-groupoid can be thought of as a choice of a differentiable atlas on a differentiable 2-stack.*

## 2.2 Generalized morphisms of Lie 2-groupoids

Generalized morphisms of Lie 2-groupoids are a straightforward generalization of generalized morphisms of Lie (1-)groupoids [24, 34]. They also have been considered in [42]. Let  $\mathbf{2Gpd}$  denote the category of Lie 2-groupoids and morphisms of Lie 2-groupoids. A *generalized morphism*  $F$  is a zigzag

$$\mathbf{\Gamma} \xleftarrow{\sim} \mathbf{E}_1 \rightarrow \dots \xleftarrow{\sim} \mathbf{E}_n \rightarrow \mathbf{\Delta},$$

where all leftward arrows are Morita morphisms. We use a squig arrow  $F : \mathbf{\Gamma} \rightsquigarrow \mathbf{\Delta}$  to denote a generalized morphism. The composition of two generalized morphisms is defined by the concatenation of two zigzags. In fact we are interested in equivalence classes of generalized morphisms:

In the sequel, we will consider two morphisms of 2-groupoids  $f : \mathbf{\Gamma} \rightarrow \mathbf{\Delta}$  and  $g : \mathbf{\Gamma} \rightarrow \mathbf{\Delta}$  to be *equivalent* if there exists two smooth applications  $\varphi : \Gamma_0 \rightarrow \Delta_1$  and  $\psi : \Gamma_1 \rightarrow \Delta_2$  such that, for any  $x \in \Gamma_2$  and any pair of composable arrows  $i, j \in \Gamma_1$ , the following relations are satisfied:

$$(g_2(x) * 1_{\varphi(s(x))}) \star \psi(l(x)) = \psi(u(x)) \star (1_{\varphi(t(x))} * f_2(x)), \quad (4)$$

$$\psi(j * i) = (1_{g_1(j)} * \psi(i)) \star (\psi(j) * 1_{f_1(i)}). \quad (5)$$

In other words,  $f$  and  $g$  are “conjugate” by a (invertible) map  $\psi$  compatible with the horizontal multiplication.

It is easy to check that the conditions (4) and (5) are equivalent to the data of a natural 2-transformation from  $f$  to  $g$  [8, 30]. Recall that a natural 2-transformation is given by the following data: an arrow  $\varphi(m) \in \Delta_1$  for each object  $m \in \Gamma_0$ , and a 2-arrow  $\psi(\gamma) \in \Delta_2$  for each arrow  $\gamma \in \Gamma_1$  as in the diagram

$$\begin{array}{ccc} f(s(j)) & \xrightarrow{\varphi(s(j))} & g(s(j)) \\ f(j) \downarrow & \nearrow \psi(j) & \downarrow g(j) \\ f(t(j)) & \xrightarrow{\varphi(t(j))} & g(t(j)) \end{array}$$

and satisfying obvious compatibility conditions with respect to the compositions of arrows and 2-arrows.

We now introduce the notion of *equivalence of generalized morphisms*; it is the natural equivalence relation on generalized morphism extending the equivalence of groupoids morphisms. Namely, we consider the weakest equivalence relation satisfying the following three properties:

- (a) If there exists a natural transformation between a pair  $f, g$  of homomorphisms of 2-groupoids,  $f$  and  $g$  are equivalent as generalized morphisms.
- (b) If  $\Gamma \xrightarrow{\phi} \Delta$  is a Morita morphism of 2-groupoids, the generalized morphisms  $\Delta \xleftarrow{\phi} \Gamma \xrightarrow{\phi} \Delta$  and  $\Gamma \xrightarrow{\phi} \Delta \xleftarrow{\phi} \Gamma$  are equivalent to  $\Delta \xrightarrow{\text{id}} \Delta$  and  $\Gamma \xrightarrow{\text{id}} \Gamma$ , respectively.
- (c) Pre- and post-composition with a third generalized morphism preserves the equivalence.

**Example 2.9.** Let  $F_1 : \Gamma \xleftarrow{\phi_1} \mathbf{E}_1 \xrightarrow{f_1} \Delta$  and  $F_2 : \Gamma \xleftarrow{\phi_2} \mathbf{E}_2 \xrightarrow{f_2} \Delta$  be two generalized morphisms. Suppose that there exists a morphism  $\mathbf{E}_1 \xrightarrow{\varepsilon} \mathbf{E}_2$  such that the diagram

$$\begin{array}{ccccc} & & \mathbf{E}_1 & & \\ & \swarrow \phi_1 & \downarrow \varepsilon & \searrow f_1 & \\ \Gamma & & & & \Delta \\ & \nwarrow \phi_2 & \uparrow & \nearrow f_2 & \\ & & \mathbf{E}_2 & & \end{array}$$

commutes up to 2-transformations. Then  $F_1$  and  $F_2$  are equivalent generalized morphisms.

**Example 2.10.** By its very definition, a Morita equivalence of groupoids  $\Gamma \xleftarrow{\sim} \mathbf{E}_1 \xrightarrow{\sim} \dots \xleftarrow{\sim} \mathbf{E}_n \xrightarrow{\sim} \Delta$  defines two generalized morphisms  $F : \Gamma \rightsquigarrow \Delta$  and  $G : \Delta \rightsquigarrow \Gamma$ . The compositions  $F \circ G$  and  $G \circ F$  are both equivalent to the identity. Furthermore, the equivalence classes of  $F$  and  $G$  are independent of the choice of the Morita equivalence.

**Remark 2.11.** Roughly speaking, generalized morphisms are obtained by formally inverting the Morita morphisms. In fact, the following Lemma is easy to check.

**Lemma 2.12.** The collection  $\mathcal{M}$  of all Morita morphisms of 2-groupoids is a left multiplicative system [28, Definition 7.1.5; 41, Definition 10.3.4] in  $\mathbf{2Gpd}$ . Indeed, the following properties hold:

- (a)  $(\Gamma \xrightarrow{\text{id}} \Gamma) \in \mathcal{M}, \forall \Gamma \in \mathbf{2Gpd}$ ;
- (b)  $\mathcal{M}$  is closed under composition;
- (c) given  $\Gamma \xrightarrow{f} \Delta \xleftarrow{\phi} \mathbf{E}$  in  $\mathbf{2Gpd}$  with  $\phi \in \mathcal{M}$ , there exists  $\Gamma \xleftarrow{\psi} \mathbf{Z} \xrightarrow{g} \mathbf{E}$  in  $\mathbf{2Gpd}$  with  $\psi \in \mathcal{M}$  such that

$$\begin{array}{ccccc} & & \mathbf{Z} & & \\ & \swarrow \psi & \searrow g & & \\ \Gamma & & & & \mathbf{E} \\ & \searrow f & \swarrow \phi & & \\ & & \Delta & & \end{array}$$

commutes;

- (d) given  $\Gamma \xrightarrow[\sim]{\phi} \Delta \xrightarrow[\sim]{f} \mathbf{E}$  in  $\mathbf{2Gpd}$  with  $\phi \in \mathcal{M}$ ,  $f \circ \phi = g \circ \phi$  implies  $f = g$ .



Since  $\mathcal{M}$  is a left multiplicative system in the category  $\mathbf{2Gpd}$ , we can consider the localization  $\mathbf{2Gpd}_{\mathcal{M}}$  of  $\mathbf{2Gpd}$  with respect to  $\mathcal{M}$  [28, Chapter 7; 41, Section 10.3]. This new category  $\mathbf{2Gpd}_{\mathcal{M}}$  has the same objects as  $\mathbf{2Gpd}$  but its arrows are equivalence classes of generalized morphisms. An isomorphism in  $\mathbf{2Gpd}_{\mathcal{M}}$  corresponds to (the equivalence class of) a Morita equivalence in  $\mathbf{2Gpd}$ .

Lemma 2.12(C) implies that any generalized morphism can be represented by a chain of length 2:

**Lemma 2.13.** *Any generalized morphism between two Lie 2-groupoids  $\Gamma$  and  $\Delta$  is equivalent to a diagram*

$$\Gamma \xleftarrow{\phi} \mathbf{E} \xrightarrow{f} \Delta$$

*in the category  $\mathbf{2Gpd}$  such that  $\phi$  is a Morita morphism (i.e.  $\phi \in \mathcal{M}$ ).*

**Remark 2.14.** *There is a bijection between maps of differentiable 2-stacks and equivalence classes of generalized morphisms of Lie 2-groupoids up to Morita equivalences.*

### 2.3 2-group bundles

In this section, we give a definition of 2-group bundles of a global nature and formulated in terms of generalized morphisms of Lie 2-groupoids.

**Definition 2.15.** *A principal (2-group)  $[G \rightarrow H]$ -bundle over a Lie groupoid  $\Gamma_1 \rightrightarrows \Gamma_0$  is a generalized morphism  $\mathfrak{B}$  from  $\Gamma_1 \rightrightarrows \Gamma_0$  (seen as a Lie 2-groupoid) to the 2-group  $[G \rightarrow H]$  associated to the crossed module  $(G \rightarrow H)$ .*

In particular, a principal  $[G \rightarrow \text{Aut}(G)]$ -bundle over a groupoid  $\Gamma_1 \rightrightarrows \Gamma_0$  is a generalized morphism from  $\Gamma_1 \rightrightarrows \Gamma_0$  (seen as a 2-groupoid) to the 2-group  $[G \rightarrow \text{Aut}(G)]$ .

Two principal  $[G \rightarrow H]$ -bundles  $\mathfrak{B}$  and  $\mathfrak{B}'$  over the groupoid  $\Gamma_1 \rightrightarrows \Gamma_0$  are said to be *isomorphic* if, and only if, these two generalized morphisms are equivalent.

A  $[G \rightarrow H]$ -bundle over a manifold  $M$  is a (2-group)  $[G \rightarrow H]$ -bundle over the groupoid  $M \rightrightarrows M$ .

Let  $\mathfrak{B}$  be a  $[G \rightarrow H]$ -bundle over a Lie groupoid  $\Gamma_1 \rightrightarrows \Gamma_0$ . If  $\Gamma'_1 \rightrightarrows \Gamma'_0$  and  $[G' \rightarrow H']$  are Morita equivalent to  $\Gamma_1 \rightrightarrows \Gamma_0$  and  $[G \rightarrow H]$  respectively, then the composition

$$(\Gamma'_1 \rightrightarrows \Gamma'_0) \rightleftarrows (\Gamma_1 \rightrightarrows \Gamma_0) \overset{\mathfrak{B}}{\rightsquigarrow} [G \rightarrow H] \rightleftarrows [G' \rightarrow H'].$$

defines a principal  $[G' \rightarrow H']$ -bundle over  $\Gamma'_1 \rightrightarrows \Gamma'_0$  denoted  $\mathfrak{B}$  by abuse of notation. Here the left and right squig arrows are the Morita equivalences seen as invertible generalized morphisms as in Example 2.10.

**Definition 2.16.** *A principal (2-group)  $[G \rightarrow H]$ -bundle  $\mathfrak{B}$  over a Lie groupoid  $\Gamma_1 \rightrightarrows \Gamma_0$  and a principal (2-group)  $[G' \rightarrow H']$ -bundle  $\mathfrak{B}'$  over a Lie groupoid  $\Gamma'_1 \rightrightarrows \Gamma'_0$  are said to be Morita equivalent if, and only if,  $\mathfrak{B}$  (viewed as a generalized morphism  $(\Gamma_1 \rightrightarrows \Gamma_0) \rightsquigarrow [G \rightarrow H]$ ) and  $\mathfrak{B}'$  are equivalent generalized morphisms.*

In particular, if  $\mathfrak{B}$  and  $\mathfrak{B}'$  are Morita equivalent,  $\Gamma_1 \rightrightarrows \Gamma_0$  and  $[G \rightarrow H]$  are Morita equivalent to  $\Gamma'_1 \rightrightarrows \Gamma'_0$  and  $[G' \rightarrow H']$  respectively.

**Remark 2.17.** When the groupoid is just a manifold, our definition is equivalent to the usual definition of 2-group bundles in [4,6,36] as suggested by Examples 2.18 and 2.19 below.

**Example 2.18.** Let  $P \xrightarrow{\pi} M$  be a principal  $H$ -bundle. Then the diagram

$$\begin{array}{ccccc}
M & \longleftarrow & P \times_M P & \longrightarrow & H \\
\Downarrow & & \Downarrow & & \Downarrow \\
M & \xleftarrow{\phi} & P \times_M P & \xrightarrow{f} & H \\
\Downarrow & & \downarrow t \quad \downarrow s & & \Downarrow \\
M & \xleftarrow{\pi} & P & \longrightarrow & *
\end{array}$$

where  $s(x, y) = x$ ,  $t(x, y) = y$ ,  $\pi(x) = \phi(x, y) = \pi(y)$  and  $x \cdot f(x, y) = y$ , defines a generalized morphism from the manifold  $M$  and to the 2-group  $[1 \rightarrow H]$ . Hence, it is a 2-group bundle over  $M$ . Note that a principal  $H$ -bundle  $P$  over  $M$  is Morita equivalent (as a 2-group bundle) to a principal  $H'$ -bundle  $P'$  over  $M'$  if, and only if,  $H$  and  $M$  are isomorphic to  $H'$  and  $M'$  respectively and  $P$  and  $P'$  are isomorphic principal bundles.

**Example 2.19.** Let  $M$  be a smooth manifold and  $G$  be a (non abelian) Lie group. A non abelian 2-cocycle [15,16,21,33] on  $M$  with values in  $G$  relative to an open covering  $\{U_i\}_{i \in I}$  of  $M$  is a collection of smooth maps

$$\lambda_{ij} : U_{ij} \rightarrow \text{Aut}(G) \quad \text{and} \quad g_{ijk} : U_{ijk} \rightarrow G$$

satisfying the following relations:

$$\begin{aligned}
\lambda_{ij} \circ \lambda_{jk} &= \mathbf{Ad}_{g_{ijk}} \circ \lambda_{ik} \\
g_{ijl} g_{jkl} &= g_{ikl} \lambda_{kl}^{-1}(g_{ijk}).
\end{aligned}$$

Such a non abelian 2-cocycle defines a  $[G \rightarrow \text{Aut}(G)]$ -bundle over the manifold  $M$ ; for it can be seen as the generalized morphism

$$\begin{array}{ccccc}
M & \longleftarrow & \coprod_{i,j} U_{ij} \times G \times G & \xrightarrow{f} & G \ltimes \text{Aut}(G) \\
\Downarrow & & \downarrow u \quad \downarrow l & & \Downarrow \\
M & \xleftarrow{\phi} & \coprod_{i,j} U_{ij} \times G & \longrightarrow & \text{Aut}(G) \\
\Downarrow & & \Downarrow & & \Downarrow \\
M & \longleftarrow & \coprod_i U_i & \longrightarrow & *
\end{array}$$

between the manifold  $M$  and the 2-group  $[G \rightarrow \text{Aut}(G)]$ . Here

$$\begin{aligned}
l(x_{ij}, g_1, g_2) &= (x_{ij}, g_1) & \phi(x_{ij}, g) &= x \\
u(x_{ij}, g_1, g_2) &= (x_{ij}, g_2) & f(x_{ij}, g_1, g_2) &= (g_2 g_1^{-1}, \mathbf{Ad}_{g_1} \circ \lambda_{ij}(x))
\end{aligned}$$

where  $x_{ij}$  denotes a point  $x \in M$  seen as a point of the open subset  $U_{ij} = U_i \cap U_j$ ,  $x_i$  the point  $x \in M$  seen as a point of the open subset  $U_i$ , and  $g, g_1, g_2$  arbitrary elements of  $G$ . The horizontal and vertical multiplication are given by

$$\begin{aligned}
(x_{ij}, \alpha) * (x_{jk}, \beta) &= (x_{ik}, g_{ijk} \lambda_{jk}^{-1}(\alpha) \beta), \\
(x_{ij}, g_1, g_2) \star (x_{ij}, g_2, g_3) &= (x_{ij}, g_1, g_3).
\end{aligned}$$

**Example 2.20.** Let  $\{U_i\}_{i \in I}$  be an open covering of a smooth manifold  $M$ . A family of smooth maps  $g_{ijk} : U_{ijk} \rightarrow S^1$  defines a Lie groupoid structure on  $\coprod_{i,j} U_{ij} \times S^1 \rightrightarrows \coprod_i U_i$  with multiplication

$$(x_{ij}, e^{i\varphi}) \cdot (x_{jk}, e^{i\psi}) = (x_{ik}, g_{ijk} e^{i(\varphi+\psi)})$$

if, and only if,  $g_{ijk}$  is a Čech 2-cocycle. In that case, we get the generalized morphism of 2-groupoids

$$\begin{array}{ccccc} M & \longleftarrow & \coprod_{i,j} U_{ij} \times S^1 \times S^{\mathfrak{f}} & \longrightarrow & S^1 \\ \Downarrow & & \downarrow u \quad \downarrow l & & \Downarrow \\ M & \longleftarrow & \coprod_{i,j} U_{ij} \times S^1 & \longrightarrow & * \\ \Downarrow & & \Downarrow & & \Downarrow \\ M & \longleftarrow & \coprod_i U_i & \longrightarrow & * \end{array}$$

with  $f(x_{ij}, e^{i\varphi}, e^{i\psi}) = e^{i(\psi-\varphi)}$ . It defines an  $[S^1 \rightarrow *]$ -bundle over  $M$ .

**Remark 2.21.** There is a nerve functor from Lie 2-groupoids to simplicial spaces generalizing the nerve for Lie 1-groupoids. For instance, see [13, 37, 40] and Section 4.1 below. Composing with (fat) realization functor, we obtain the classifying space functor  $\mathbf{\Gamma} \rightarrow B\mathbf{\Gamma}$  from Lie 2-groupoids to topological spaces. Since the realization of a Morita morphism is an homotopy equivalence, a generalized morphism  $\mathbf{\Gamma} \xrightarrow{F} \mathbf{\Delta}$  induces a map  $B\mathbf{\Gamma} \xrightarrow{BF} B\mathbf{\Delta}$  in the homotopy category of topological spaces. In particular, a  $[G \rightarrow H]$ -group bundle over a manifold induces a map  $M \rightarrow B[G \rightarrow H]$  in the homotopy category. This is the topological side of generalized morphism and 2-group bundles. In fact, using standard arguments on homotopy for manifolds, it should be possible to prove that  $[G \rightarrow H]$ -group bundles over  $\mathbf{\Gamma}$  (up to Morita equivalences) are in bijection with homotopy classes of maps  $B\mathbf{\Gamma} \rightarrow B[G \rightarrow H]$ .

### 3 Groupoid $G$ -extensions

We fix a Lie group  $G$ .

**Definition 3.1.** A Lie groupoid  $G$ -extension is a short exact sequence of Lie groupoids over the identity map on the unit space  $M$

$$1 \rightarrow M \times G \xrightarrow{i} \tilde{\Gamma} \xrightarrow{\phi} \Gamma \rightarrow 1 \quad (6)$$

Here both  $\Gamma$  and  $\tilde{\Gamma}$  are Lie groupoids over  $M$  and  $M \times G \rightrightarrows M$  is a (trivial) bundle of groups.

In the sequel, an extension like (6) will be denoted  $\tilde{\Gamma} \xrightarrow{\phi} \Gamma \rightrightarrows M$  and we will write  $g_m$  instead of  $i(m, g)$ .

In terms of crossed modules, Lie groupoid  $G$ -extensions can be seen as follows.

**Proposition 3.2.** The morphism of groupoids  $\tilde{\Gamma} \xrightarrow{\phi} \Gamma \rightrightarrows M$  is a groupoid  $G$ -extension if, and only if,  $(M \times G \hookrightarrow \tilde{\Gamma})$  is a crossed module of groupoids with quotient groupoid  $\tilde{\Gamma}/i(M \times G)$  isomorphic to  $\Gamma$ .

The proof of Proposition follows easily from Remark 3.7 and Lemma 3.9 below.

**Definition 3.3** ([26]). *A homomorphism of Lie groupoid  $G$ -extensions*

$$\begin{array}{ccccc} \tilde{\Gamma} & \longrightarrow & \Gamma & \rightrightarrows & M \\ f \downarrow & & f \downarrow & & f \downarrow \\ \tilde{\Delta} & \longrightarrow & \Delta & \rightrightarrows & N \end{array}$$

is a Morita morphism if  $M \xrightarrow{f} N$  is a surjective submersion and

$$\begin{array}{ccc} \Gamma & \xrightarrow{f} & \Delta \\ \Downarrow & & \Downarrow \\ M & \xrightarrow{f} & N \end{array} \quad \text{and} \quad \begin{array}{ccc} \tilde{\Gamma} & \xrightarrow{f} & \tilde{\Delta} \\ \Downarrow & & \Downarrow \\ M & \xrightarrow{f} & N \end{array}$$

are Morita morphisms of 1-groupoids.

As in the Lie 2-groupoid case, the Morita morphisms of Lie groupoid extensions form a left multiplicative system in the category of Lie groupoid extensions and homomorphisms of Lie groupoid extensions. Hence, one can localize this category by its Morita morphisms. Two Lie groupoid extensions are Morita equivalent if they are isomorphic in the localized category. (As in the Lie 2-groupoids case, there is a notion of generalized morphisms for Lie groupoid extensions. In that language, a Morita equivalence is an invertible generalized morphism.)

Here is our first main theorem.

**Theorem 3.4.** *There exists a bijection between the Morita equivalence classes of Lie groupoid  $G$ -extensions and the Morita equivalence classes of  $[G \rightarrow \text{Aut}(G)]$ -bundles over Lie groupoids.*

**Remark 3.5.** *The above theorem can be regarded as a geometric version of a theorem of Breen [9], which states that  $H^2(\mathfrak{X}, G)$  is isomorphic to  $H^1(\mathfrak{X}, (G \rightarrow \text{Aut}(G)))$ .*

The proof of Theorem 3.4 is the object of the next two sections.

### 3.1 From groupoid $G$ -extensions to $[G \rightarrow \text{Aut}(G)]$ -bundles

Given a Lie groupoid  $G$ -extension  $\tilde{\Gamma} \xrightarrow{\phi} \Gamma \xrightleftharpoons[b]{a} M$ , one can define a Lie 2-

groupoid  $\tilde{\Gamma} \times_{\Gamma} \tilde{\Gamma} \xrightleftharpoons[u]{l} \tilde{\Gamma} \xrightleftharpoons[t]{s} M$ , where

$$\begin{aligned} \tilde{\Gamma} \times_{\Gamma} \tilde{\Gamma} &= \{(\tilde{\gamma}_1, \tilde{\gamma}_2) \in \tilde{\Gamma} \times \tilde{\Gamma} \mid \phi(\tilde{\gamma}_1) = \phi(\tilde{\gamma}_2)\} \\ l(\tilde{\gamma}_1, \tilde{\gamma}_2) &= \tilde{\gamma}_2 & u(\tilde{\gamma}_1, \tilde{\gamma}_2) &= \tilde{\gamma}_1 \\ s(\tilde{\gamma}) &= a(\phi(\tilde{\gamma})) & t(\tilde{\gamma}) &= b(\phi(\tilde{\gamma})) \\ (\tilde{\gamma}_1, \tilde{\gamma}_2) \star (\tilde{\gamma}_2, \tilde{\gamma}_3) &= (\tilde{\gamma}_1, \tilde{\gamma}_3) \\ (\tilde{\gamma}_1, \tilde{\gamma}_2) * (\tilde{\delta}_1, \tilde{\delta}_2) &= (\tilde{\gamma}_1 \cdot \tilde{\delta}_1, \tilde{\gamma}_2 \cdot \tilde{\delta}_2). \end{aligned}$$

Here  $\cdot$  stands for the multiplication in  $\tilde{\Gamma} \rightrightarrows M$ .

The groupoid homomorphism  $\phi$  naturally induces a Morita morphism of 2-groupoids:

$$\begin{array}{ccc}
 \tilde{\Gamma} \times_{\Gamma} \tilde{\Gamma} & \longrightarrow & \Gamma \\
 \begin{array}{c} \Downarrow u \\ \Downarrow l \end{array} & & \begin{array}{c} \Downarrow \text{id} \\ \Downarrow \text{id} \end{array} \\
 \tilde{\Gamma} & \xrightarrow{\phi} & \Gamma \\
 \begin{array}{c} \Downarrow t \\ \Downarrow s \end{array} & & \begin{array}{c} \Downarrow b \\ \Downarrow a \end{array} \\
 M & \xrightarrow{\text{id}} & M
 \end{array} \tag{7}$$

where the 2-groupoid  $\Gamma \xrightarrow[\text{id}]{\text{id}} \Gamma \xrightarrow[t]{s} M$  is simply the groupoid  $\Gamma \xrightarrow[t]{s} M$  seen as a 2-groupoid in the trivial way.

Consider the map  $\tilde{\Gamma} \rightarrow \text{Aut}(G) : \tilde{\gamma} \mapsto \mathbf{Ad}_{\tilde{\gamma}}$  defined by  $(\mathbf{Ad}_{\tilde{\gamma}} g)_{t(\tilde{\gamma})} = \tilde{\gamma} \cdot g_{s(\tilde{\gamma})} \cdot \tilde{\gamma}^{-1}$ . It gives a morphism of Lie groupoids

$$\begin{array}{ccc}
 \tilde{\Gamma} & \xrightarrow{\mathbf{Ad}} & \text{Aut}(G) \\
 \Downarrow & & \Downarrow \\
 M & \longrightarrow & *
 \end{array} \tag{8}$$

which, together with the map

$$\tilde{\Gamma} \times_{\Gamma} \tilde{\Gamma} \rightarrow G \ltimes \text{Aut}(G) : (\tilde{\gamma}_1, \tilde{\gamma}_2) \mapsto (g, \mathbf{Ad}_{\tilde{\gamma}_2}),$$

where  $\tilde{\gamma}_1 \tilde{\gamma}_2^{-1} = g_{t(\tilde{\gamma}_1)}$ , defines a homomorphism of Lie 2-groupoids

$$\begin{array}{ccc}
 \tilde{\Gamma} \times_{\Gamma} \tilde{\Gamma} & \longrightarrow & G \ltimes \text{Aut}(G) \\
 \Downarrow & & \Downarrow \\
 \tilde{\Gamma} & \xrightarrow{\mathbf{Ad}} & \text{Aut}(G) \\
 \Downarrow & & \Downarrow \\
 M & \longrightarrow & *
 \end{array} \tag{9}$$

**Remark 3.6.** Note that the induced map

$$\begin{array}{ccc}
 \tilde{\Gamma} \times_{\Gamma} \tilde{\Gamma} & \longrightarrow & G \ltimes \text{Aut}(G) \\
 \Downarrow & & \Downarrow \\
 \tilde{\Gamma} & \longrightarrow & \text{Aut}(G)
 \end{array}$$

is a fully faithful functor.

**Remark 3.7.** In terms of crossed modules, the above discussion goes as follows. The extension  $1 \rightarrow M \times G \xrightarrow{i} \tilde{\Gamma} \xrightarrow{\phi} \Gamma \rightarrow 1$  leads to an action of  $\tilde{\Gamma}$  on the groupoid  $M \times G \rightrightarrows M$  by conjugation, i.e. via the map  $\tilde{\gamma} \mapsto \mathbf{Ad}_{\tilde{\gamma}}$ . Then  $\tilde{\Gamma} \times_{\Gamma} \tilde{\Gamma} \xrightarrow[u]{l} \tilde{\Gamma} \xrightarrow[t]{s} M$  is the Lie 2-groupoid corresponding to the crossed module  $(M \times G \xrightarrow{i} \tilde{\Gamma})$ . The projection onto the first factor  $M \times G \rightarrow M$  and the morphism  $\phi : \tilde{\Gamma} \rightarrow \Gamma$  induce the Morita equivalence of crossed modules  $(M \times G \rightarrow \tilde{\Gamma}) \rightarrow (M \rightarrow \Gamma)$  corresponding to the map (7). Moreover, the map  $\mathbf{Ad} : \tilde{\Gamma} \rightarrow \text{Aut}(G)$  yields the map of crossed modules  $(M \times G \xrightarrow{i} \tilde{\Gamma}) \xrightarrow{(\text{id}, \mathbf{Ad})} (G \rightarrow \text{Aut}(G))$  corresponding to the morphism of Lie 2-groupoids (9).

**Proposition 3.8.** (a) A Lie groupoid  $G$ -extension  $\tilde{\Gamma} \rightarrow \Gamma \rightrightarrows M$  induces a principal  $[G \rightarrow \text{Aut}(G)]$ -bundle over  $\Gamma \rightrightarrows M$ , which can be described explicitly by the following generalized morphism:

$$\begin{array}{ccccc} \Gamma & \longleftarrow & \tilde{\Gamma} \times_{\Gamma} \tilde{\Gamma} & \longrightarrow & G \ltimes \text{Aut}(G) \\ \Downarrow & & \Downarrow & & \Downarrow \\ \Gamma & \longleftarrow & \tilde{\Gamma} & \longrightarrow & \text{Aut}(G) \\ \Downarrow & & \Downarrow & & \Downarrow \\ M & \longleftarrow & M & \longrightarrow & * \end{array}$$

(b) If  $\tilde{\Gamma} \rightarrow \Gamma \rightrightarrows M$  and  $\tilde{\Delta} \rightarrow \Delta \rightrightarrows N$  are Morita equivalent  $G$ -extensions, then the corresponding 2-group bundles are Morita equivalent.

*Proof.* Claim (a) follows from the above discussion. Suppose given a Morita morphism of  $G$ -extensions

$$\begin{array}{ccccc} \tilde{\Gamma} & \longrightarrow & \Gamma & \rightrightarrows & M \\ f \downarrow & & f \downarrow & & f \downarrow \\ \tilde{\Delta} & \longrightarrow & \Delta & \rightrightarrows & N. \end{array}$$

Since  $f$  commutes with the  $\tilde{\Gamma}$  and  $\tilde{\Delta}$ -actions on  $G$ , there is a commutative diagram

$$\begin{array}{ccc} [M \rightarrow \Gamma] & \xleftarrow{\sim} & [M \times G \rightarrow \tilde{\Gamma}] \xrightarrow{(p_2, \mathbf{Ad})} [G \rightarrow \text{Aut}(G)] \\ (f, f) \downarrow \sim & & (f \times p_2, f) \downarrow \sim \nearrow (p_2, \mathbf{Ad}) \\ [N \rightarrow \Delta] & \xleftarrow{\sim} & [N \times G \rightarrow \tilde{\Delta}] \end{array}$$

Now, claim (b) follows from Example 2.9. □

### 3.2 From $[G \rightarrow \text{Aut}(G)]$ -bundles to groupoid $G$ -extensions

**Lemma 3.9.** *Let*

$$\begin{array}{ccc} \Delta_2 & \xrightarrow{\phi_2} & \Gamma_2 \\ \Downarrow & & \Downarrow \\ \Delta_1 & \xrightarrow{\phi_1} & \Gamma_1 \\ \Downarrow & & \Downarrow \\ \Delta_0 & \xrightarrow{\phi_0} & \Gamma_0 \end{array}$$

*be a Morita morphism of 2-groupoids. And let*

$$\begin{array}{ccc} L & \xrightarrow{\phi} & K \\ j \downarrow & & \downarrow i \\ \Delta_1 & \xrightarrow{\phi} & \Gamma_1 \end{array}$$

*be the induced map of crossed modules. Then  $\phi$  maps  $i^{-1}(1_m)$  onto  $j^{-1}(1_{\phi(m)})$  bijectively and induces a functor from the groupoid  $\frac{\Delta_1}{j(L)}$  to the groupoid  $\frac{\Gamma_1}{i(K)}$ , which is fully faithful and surjective on the objects.*

The crossed modules  $[1 \rightarrow \frac{\Delta_1}{j(L)}]$  and  $[1 \rightarrow \frac{\Gamma_1}{i(K)}]$  are usually called the cokernel of the crossed modules  $[L \xrightarrow{j} \Delta_1]$  and  $[K \xrightarrow{i} \Gamma_1]$  respectively.

**Proposition 3.10.** (a) A  $[G \rightarrow \text{Aut}(G)]$ -bundle over a Lie groupoid  $\Gamma \rightrightarrows \Gamma_0$  induces a Lie groupoid  $G$ -extension.

(b) Morita equivalent  $[G \rightarrow \text{Aut}(G)]$ -bundles induce Morita equivalent extensions.

*Proof.* (a) Suppose the  $[G \rightarrow \text{Aut}(G)]$ -bundle is given by the generalized morphism of 2-groupoids

$$\begin{array}{ccccc} \Gamma & \xleftarrow{\phi} & \Delta_2 & \xrightarrow{f} & G \ltimes \text{Aut}(G) \\ \Downarrow & & \Downarrow & & \Downarrow \\ \Gamma & \xleftarrow{\phi} & \Delta_1 & \xrightarrow{f} & \text{Aut}(G) \\ \Downarrow & & \Downarrow & & \Downarrow \\ \Gamma_0 & \xleftarrow{\phi} & \Delta_0 & \xrightarrow{f} & * \end{array}$$

and let

$$\begin{array}{ccccc} \Gamma_0 & \xleftarrow{\phi} & L & \xrightarrow{f} & G \\ \downarrow i & & \downarrow j & & \downarrow \text{Ad} \\ \Gamma & \xleftarrow{\phi} & \Delta_1 & \xrightarrow{f} & \text{Aut}(G) \end{array}$$

be the induced map of crossed modules. Since  $\phi$  is a Morita morphism and  $\Gamma_0 \xrightarrow{i} \Gamma$  is an injection, by Lemma 3.9,  $L \xrightarrow{j} \Delta_1$  is also injective and

$$\begin{array}{ccc} \frac{\Delta_1}{j(L)} & \xrightarrow{\phi} & \frac{\Gamma}{i(\Gamma_0)} \\ \Downarrow & & \Downarrow \\ \Delta_0 & \xrightarrow{\phi} & \Gamma_0 \end{array}$$

is a fully faithful functor. Since  $\Delta_0 \xrightarrow{\phi} \Gamma_0$  is a surjective submersion and  $\frac{\Gamma}{i(\Gamma_0)}$  is diffeomorphic to  $\Gamma$ , we obtain a Morita morphism of Lie 1-groupoids from  $H \rightrightarrows \Delta_0$  to  $\Gamma \rightrightarrows \Gamma_0$ , where  $H$  is, by definition, the smooth manifold  $\frac{\Delta_1}{j(L)}$ . The bundle of groups  $L$  can be embedded into  $\Delta_1 \times G$  as a “normal” subgroupoid using the map  $l \mapsto (j(l), f(l))$ . The quotient (by  $j(L)$ )  $\tilde{H} \rightarrow H \rightrightarrows \Delta_0$  of the trivial extension  $\Delta_1 \times G \rightarrow \Delta_1 \rightrightarrows \Delta_0$  is a  $G$ -extension of groupoids. Note that its corresponding crossed module is  $(\Delta_0 \times G \rightarrow \tilde{H})$ . (b) It is sufficient to check that for any diagram

$$\begin{array}{ccccc} & & \mathbf{E} & & \\ \phi_1 \swarrow & & \downarrow \varepsilon & \searrow f_1 & \\ \Gamma & & & & [G \xrightarrow{\text{Ad}} \text{Aut}(G)] \\ \phi_2 \swarrow & & \downarrow \varepsilon & \searrow f_2 & \\ & & \mathbf{F} & & \end{array}$$

commuting up to natural 2-equivalences, the  $G$ -extension corresponding to the lower and upper generalized morphism are Morita equivalent. Since  $\phi_1, \phi_2$  are Morita equivalence,  $\varepsilon$  is also Morita equivalence. Therefore, by Lemma 2.13, we

can assume that  $\varepsilon$  is a Morita morphism. Then, denoting by  $(K \rightarrow E)$  and  $(L \rightarrow F)$  the crossed modules corresponding to  $\mathbf{E}$  and  $\mathbf{F}$  respectively, the map  $\varepsilon$  induces a commutative diagram

$$\begin{array}{ccccc} E_1 \times_{j(K)} G & \longrightarrow & E_1 & \rightrightarrows & E_0 \\ (\varepsilon, \text{id}) \downarrow & & \varepsilon \downarrow & & \varepsilon \downarrow \\ F_1 \times_{j(L)} G & \longrightarrow & F_1 & \rightrightarrows & F_0 \end{array}$$

which is a Morita equivalence of extensions by Lemma 3.9.  $\square$

### 3.3 Proof of Theorem 3.4

It remains to prove that the constructions of Section 3.1 and Section 3.2 are inverse of each other.

Suppose that a  $[G \rightarrow \text{Aut}(G)]$ -principal bundle  $\mathfrak{B}$  is given by the generalized morphism  $\Gamma \xleftarrow{\phi} \Delta \xrightarrow{f} [G \rightarrow \text{Aut}(G)]$ . Let  $\tilde{H} \rightarrow H \rightrightarrows \Delta_0$  be the induced  $G$ -principal extension. As in the proof of Proposition 3.10, we denote  $(L \rightarrow \Delta_1)$  the crossed module corresponding to  $\Delta$ . Then, the crossed module corresponding to  $\tilde{H}$  is  $(\Delta_0 \times G \rightarrow \tilde{H})$ .

Therefore, we have the following commutative diagram of crossed modules:

$$\begin{array}{ccccc} (\Gamma_0 \rightarrow \Gamma) & \xleftarrow{\sim} & (L \rightarrow \Delta_1) & \longrightarrow & (G \rightarrow \text{Aut}(G)) \\ \sim \uparrow & & \downarrow & \nearrow & \\ (\Delta_0 \rightarrow H) & \xleftarrow{\sim} & (\Delta_0 \times G \rightarrow \tilde{H}) & & \end{array}$$

It follows that the generalized morphism

$$[\Gamma_0 \rightarrow \Gamma] \xleftarrow{\sim} [L \rightarrow \Delta_1] \rightarrow [G \rightarrow \text{Aut}(G)]$$

we started from is equivalent to the generalized morphism

$$[\Gamma_0 \rightarrow \Gamma] \xleftarrow{\sim} [\Delta_0 \times G \rightarrow \tilde{H}] \rightarrow [G \rightarrow \text{Aut}(G)]$$

associated to the  $G$ -extension  $\tilde{H} \rightarrow H \rightrightarrows \Delta_0$ . Hence they represent the same  $(G \rightarrow \text{Aut}(G))$ -bundle over  $\Gamma \rightrightarrows \Gamma_0$ .

Reciprocally, if  $\tilde{\Gamma} \rightarrow \Gamma \rightrightarrows M$  is  $G$ -extension, then the associated principal  $[G \rightarrow \text{Aut}(G)]$ -bundle is given by the generalized morphism

$$[M \rightarrow \Gamma] \xleftarrow{\phi} [M \times G \rightarrow \tilde{\Gamma}] \xrightarrow{(id, \text{Ad})} [G \rightarrow \text{Aut}(G)] \quad (10)$$

according to Remark 3.7. Direct inspection of the proof of Proposition 3.10 shows that the  $G$ -extension induced by the generalized morphism (10) is  $\tilde{\Gamma} \rightarrow \Gamma \rightrightarrows M$ .

## 4 Universal characteristic maps and Dixmier-Douady classes

### 4.1 Cohomology of Lie 2-groupoids

As in the case of groupoids, associated to a Lie 2-groupoid  $\Gamma_2 \rightrightarrows \Gamma_1 \rightrightarrows \Gamma_0$ , there is a simplicial manifold  $N_\bullet \Gamma$ , called its (geometric) nerve. It is the nerve



of the underlying 2-category as defined by Street [40]. In particular,  $N_0 \mathbf{\Gamma} = \Gamma_0$ ,  $N_1 \mathbf{\Gamma} = \Gamma_1$ ,  $N_2 \mathbf{\Gamma}$  is a submanifold of  $\Gamma_2 \times \Gamma_1 \times \Gamma_1 \times \Gamma_1$  parameterizing the 2-arrows of  $\Gamma_2$  fitting in a commutative triangle

$$\begin{array}{ccc} & A_1 & \\ f_2 \nearrow & \Downarrow \alpha & \searrow f_0 \\ A_0 & \xrightarrow{f_1} & A_2 \end{array} \quad (11)$$

and  $N_3 \mathbf{\Gamma}$  is a submanifold of  $(\Gamma_2)^4 \times (\Gamma_1)^6$  parameterizing the commutative tetrahedra like

$$\begin{array}{ccccc} & & A_3 & & \\ & f_{12} \nearrow & \Downarrow \alpha_2 & \nwarrow f_{02} & \\ & & A_1 & & \\ & f_{23} \nearrow & \Downarrow \alpha_3 & \nwarrow f_{03} & \\ A_0 & \xrightarrow{f_{13}} & A_2 & & \end{array} \quad (12)$$

with faces given by elements of  $N_2 \mathbf{\Gamma}$ . By the commutativity of the tetrahedron (12), we mean that  $(\alpha_3 * f_{01}) \star \alpha_1 = (f_{23} * \alpha_0) \star \alpha_2$ . For  $p \geq 3$ ,  $N_p \mathbf{\Gamma}$  is the manifold of all  $p$ -simplices such that each subsimplex of dimension 3 is a tetrahedron of the form (12) above [18, 40]. The nerve of a Lie groupoid considered as a Lie 2-groupoid is isomorphic to its usual (1-)nerve [37]. The nerve  $N_\bullet$  defines a functor from the category of Lie 2-groupoids to the category of simplicial manifolds. Taking the fat realization of the nerve defines a functor from Lie 2-groupoids to topological spaces.

The de Rham cohomology groups of a Lie 2-groupoid  $\mathbf{\Gamma}$  are defined to be the cohomology groups of the bicomplex  $(\Omega^\bullet(N_\bullet \mathbf{\Gamma}), d_{\text{DR}}, \partial)$ , where  $d_{\text{DR}} : \Omega^p(N_q \mathbf{\Gamma}) \rightarrow \Omega^{p+1}(N_q \mathbf{\Gamma})$  is the de Rham differential and  $\partial : \Omega^p(N_q \mathbf{\Gamma}) \rightarrow \Omega^p(N_{q+1} \mathbf{\Gamma})$  is defined by  $\partial = (-1)^p \sum_{i=0}^{q+1} (-1)^i d_i^*$ , where  $d_i : N_{q+1} \mathbf{\Gamma} \rightarrow N_q \mathbf{\Gamma}$  denotes the  $i^{\text{th}}$  face map. We use the shorter notation  $\Omega_{\text{tot}}^\bullet(\mathbf{\Gamma})$  for the associated total complex. Hence  $\Omega_{\text{tot}}^n(\mathbf{\Gamma}) = \bigoplus_{p+q=n} \Omega^p(N_q \mathbf{\Gamma})$  with (total) differential  $d_{\text{DR}} + \partial$ . We denote the subspaces of cocycles and coboundaries by  $Z_{\text{DR}}^\bullet(\mathbf{\Gamma})$  and  $B_{\text{DR}}^\bullet(\mathbf{\Gamma})$  respectively, and the cohomology of  $\mathbf{\Gamma}$  by  $H^\bullet(\mathbf{\Gamma})$ .

**Lemma 4.1.** *Let  $F : \mathbf{\Gamma} \rightarrow \mathbf{\Delta}$  be a Morita morphism of Lie 2-groupoids. Then  $F^* : H^\bullet(\mathbf{\Delta}) \rightarrow H^\bullet(\mathbf{\Gamma})$  is an isomorphism.*

*Proof.* It is well-known that a natural transformation between two 2-functors  $f$  and  $g$  from  $\mathbf{\Gamma}$  to  $\mathbf{\Delta}$  induces a simplicial homotopy between  $f_* : N_\bullet(\mathbf{\Gamma}) \rightarrow N_\bullet(\mathbf{\Delta})$  and  $g_* : N_\bullet(\mathbf{\Gamma}) \rightarrow N_\bullet(\mathbf{\Delta})$ , for instance see [13, Proposition 4]. In particular equivalent 2-categories have homotopic nerves. The result follows since a Morita morphism is an equivalence of 2-categories.  $\square$

By Lemma 4.1 above, a generalized morphism  $F : \mathbf{\Gamma} \xleftarrow[\sim]{\phi_1} \mathbf{E}_1 \xrightarrow{f_1} \dots \xleftarrow[\sim]{\phi_n} \mathbf{E}_n \xrightarrow{f_n} \mathbf{\Delta}$  induces a pullback map in cohomology

$$F^* : H^\bullet(\mathbf{\Delta}) \xrightarrow{f_n^*} H^\bullet(\mathbf{E}_n) \xrightarrow{(\phi_n^*)^{-1}} \dots \xrightarrow{f_1^*} H^\bullet(\mathbf{E}_1) \xrightarrow{(\phi_1^*)^{-1}} H^\bullet(\mathbf{\Gamma})$$

Clearly,  $(F \circ G)^* = G^* \circ F^*$  and, if  $F$  is a Morita equivalence, then  $F^*$  is an isomorphism.

**Lemma 4.2.** *If  $F$  and  $G$  are equivalent generalized morphisms from  $\Gamma$  to  $\Delta$ , the maps  $F^*$  and  $G^*$ , which they induce at the cohomology level, are equal.*

*Proof.* A natural transformation between two 2-functors  $f$  and  $g$  from  $\Gamma$  to  $\Delta$  induces a simplicial homotopy between  $f_* : N_\bullet(\Gamma) \rightarrow N_\bullet(\Delta)$  and  $g_* : N_\bullet(\Gamma) \rightarrow N_\bullet(\Delta)$  (see [13]). Therefore the lemma follows from the definition of equivalence of generalized morphisms and Lemma 4.1.  $\square$

**Remark 4.3.** *Note that, for a Lie groupoid  $\Gamma : \Gamma_2 \xrightarrow[u]{l} \Gamma_1 \xrightarrow[t]{s} \Gamma_0$ ,  $N_2 \Gamma$  may be identified to  $\Gamma_2 \times_{s, \Gamma_0, t} \Gamma_1$  so that the face maps take the form*

$$\begin{aligned} d_0 : N_2 \Gamma &\rightarrow N_1 \Gamma : (\alpha, c) \mapsto u(\alpha) \\ d_1 : N_2 \Gamma &\rightarrow N_1 \Gamma : (\alpha, c) \mapsto l(\alpha)c \\ d_2 : N_2 \Gamma &\rightarrow N_1 \Gamma : (\alpha, c) \mapsto c \end{aligned}$$

## 4.2 Cohomology characteristic map for 2-group bundles

Fix a crossed module  $G \rightarrow H$  and let  $\mathfrak{B}$  be a principal  $[G \rightarrow H]$ -bundle over  $\Gamma$ . In this section we construct a universal characteristic homomorphism  $\mathbf{CC}_{\mathfrak{B}} : H^\bullet([G \rightarrow H]) \rightarrow H^\bullet(\Gamma)$  generalizing the usual characteristic classes of a principal bundle.

By definition,  $\mathfrak{B}$  is a generalized morphism  $\Gamma \rightsquigarrow [G \rightarrow H]$ . Therefore, passing to cohomology, we obtain the homomorphism

$$\mathbf{CC}_{\mathfrak{B}} : H^\bullet([G \rightarrow H]) \xrightarrow{\mathfrak{B}^*} H^\bullet(\Gamma) \quad (13)$$

which we call *the characteristic homomorphism* of the  $[G \rightarrow H]$ -bundle  $\mathfrak{B}$ . It depends only on the isomorphism class of the 2-group bundle.

**Proposition 4.4.** *If  $\mathfrak{B}$  and  $\mathfrak{B}'$  are isomorphic  $[G \rightarrow H]$ -bundles over  $\Gamma$ , then  $\mathbf{CC}_{\mathfrak{B}} = \mathbf{CC}_{\mathfrak{B}'} : H^\bullet([G \rightarrow H]) \rightarrow H^\bullet(\Gamma)$ .*

*Proof.* It is an immediate consequence of Lemma 4.1 since isomorphic principal 2-group bundles are equivalent as generalized morphisms.  $\square$

**Remark 4.5.** *By analogy with the case of principal bundles, one can think of the elements of  $H^\bullet([G \rightarrow H])$  as universal characteristic classes and their images in  $H^\bullet(\Gamma)$  by  $\mathbf{CC}_{\mathfrak{B}}$  as characteristic classes of the  $[G \rightarrow H]$ -bundle over  $\Gamma$ .*

**Example 4.6.** *Let  $P \xrightarrow{\pi} M$  be a principal  $H$ -bundle. Then, by Example 2.18,  $P$  induces a structure of  $[1 \rightarrow H]$ -bundle over  $M$ . Since  $H^\bullet([1 \rightarrow H]) \cong H^\bullet(BH)$ , the characteristic map  $\mathbf{CC}_P$  of this bundle coincides with the classical map  $H^\bullet(BH) \rightarrow H^\bullet(M)$  induced by the principal  $H$ -bundle structure on  $P$ . In particular, for a compact Lie group  $H$ , the characteristic map coincides with the Chern-Weil map  $S(\mathfrak{h}^*)^{\mathfrak{h}} \rightarrow H^\bullet(M)$  induced by the choice of a connection for  $P$ .*

Let  $\Gamma \xrightarrow{F} \Delta$  be a generalized morphism of Lie (1-)groupoids and let  $\mathfrak{B} : \Delta \xleftarrow[\sim]{\phi} \mathbf{E} \xrightarrow{f} [G \rightarrow H]$  be a 2-group bundle with base  $\Delta$ . The pullback  $F^*(\mathfrak{B})$  of the  $[G \rightarrow H]$ -bundle  $\mathfrak{B}$  from  $\Delta$  to  $\Gamma$  by  $F$  is the composition  $\mathfrak{B} \circ F$  of the two generalized morphisms. It is a principal  $[G \rightarrow H]$ -bundle over  $\Gamma$ .

The Whitney sum of two 2-group bundles is defined as follows. Let  $\mathfrak{B} : \Gamma \xleftarrow[\sim]{\phi} \mathbf{E} \xrightarrow{f} [G \rightarrow H]$  and  $\mathfrak{B}' : \Gamma \xleftarrow[\sim]{\phi'} \mathbf{E}' \xrightarrow{f'} [G' \rightarrow H']$  be two 2-group bundles over the same base  $\Gamma$ . Let  $\mathbf{F}$  be the “fiber product” 2-groupoid  $E_2 \times_{\Gamma_2} E'_2 \rightrightarrows E_1 \times_{\Gamma_1} E'_1 \rightrightarrows E_0 \times_{\Gamma_0} E'_0$  with the obvious structure maps  $s(e, e') = (s(e), s(e'))$ ,  $(x, x') * (y, y') = (x * x', y * y')$  and so on. The Whitney sum  $\mathfrak{B} \oplus \mathfrak{B}'$  is the  $[G \times G' \rightarrow H \times H']$ -bundle over  $\Gamma$  given by the generalized morphism  $\Gamma \xleftarrow[\sim]{\phi \times \phi'} \mathbf{F} \xrightarrow{f \times f'} [G \times G' \rightarrow H \times H']$ .

By Proposition 4.4, we obtain

**Corollary 4.7.** (a)  $\mathbf{CC}_{F^*(\mathfrak{B})} = F^* \circ \mathbf{CC}_{\mathfrak{B}}$ .

(b)  $\mathbf{CC}_{\mathfrak{B} \oplus \mathfrak{B}'} = \Delta^* \circ (\mathbf{CC}_{\mathfrak{B}} \times \mathbf{CC}_{\mathfrak{B}'}),$  where  $\Delta : \Gamma \rightarrow \Gamma \times \Gamma$  is the diagonal map and  $\times$  is the cross-product  $H^\bullet([G \rightarrow H]) \otimes H^\bullet([G' \rightarrow H']) \cong H^\bullet([G \times G' \rightarrow H \times H'])$ .

**Remark 4.8.** By Proposition 3.8, a Lie groupoid  $G$ -extension  $\tilde{\Gamma} \xrightarrow{\phi} \Gamma \rightrightarrows M$  induces a principal  $[G \rightarrow \text{Aut}(G)]$ -bundle  $\mathfrak{B}_\phi$  over the groupoid  $\Gamma \rightrightarrows M$ . Hence we obtain the universal characteristic map  $\mathbf{CC}_{\mathfrak{B}_\phi} : H^\bullet([G \xrightarrow{\text{Ad}} \text{Aut}(G)]) \rightarrow H^\bullet(\Gamma)$ . Unfortunately, the cohomology  $H^\bullet([G \xrightarrow{\text{Ad}} \text{Aut}(G)])$  is not known when the center of  $G$  is large and it is trivial when the center of  $G$  is of dimension less than 3 [20]. Therefore one cannot have much hope of getting interesting characteristic classes except for extensions whose structure 2-group can be reduced. Indeed, this is the object of the next Section.

### 4.3 DD classes for groupoid central $G$ -extensions

Let  $\tilde{\Gamma} \xrightarrow{\phi} \Gamma \rightrightarrows M$  be a  $G$ -extension of Lie groupoids. Let  $\phi'$  denote the factorization of the morphism  $\phi$  through the projection  $q : \tilde{\Gamma} \rightarrow \tilde{\Gamma}/Z(G)$ :

$$\begin{array}{ccc} \tilde{\Gamma} & \xrightarrow{\phi} & \Gamma \\ q \downarrow & \nearrow \phi' & \\ \tilde{\Gamma}/Z(G) & & \end{array}$$

The extension  $\phi$  is said to be *central* [26] if there exists a section  $\sigma : \Gamma \rightarrow \tilde{\Gamma}/Z(G)$  of  $\phi'$  such that

$$xg = gx \quad \forall x \in q^{-1}(\sigma(\Gamma)), \forall g \in G \quad (14)$$

In this case, the subspace  $\tilde{\Gamma}' = q^{-1}(\sigma(\Gamma))$  of  $\tilde{\Gamma}$  is a central  $Z(G)$ -extension of  $\Gamma \rightrightarrows M$ .

Given  $\gamma \in \tilde{\Gamma}$ , there exists  $x \in \tilde{\Gamma}'$  such that  $\phi(x) = \phi(\gamma)$ . Thus there exists  $k \in G$  such that  $\gamma = x \cdot k$ . Given  $\gamma$ , both  $x$  and  $k$  are uniquely determined up to an element of  $Z(G)$ . Defining a homomorphism of Lie groupoids  $r : \tilde{\Gamma} \rightarrow G/Z(G)$  by the relation  $q(\gamma) = \sigma(\phi(\gamma))r(\gamma)$ , we obtain that, for any  $g \in G$ ,

$$g\gamma = gxk = xgk = xk \cdot k^{-1}gk = \gamma g^{r(\gamma)}$$

where  $g^{r(\gamma)}$  denotes the conjugate  $k^{-1}gk$  of  $g$  by any element  $k \in G$  such that  $kZ(G) = r(\gamma)$ .

**Proposition 4.9.** *Let  $\tilde{\Gamma} \xrightarrow{\phi} \Gamma \rightrightarrows M$  be a  $G$ -extension of a Lie groupoid  $\Gamma$  and let  $\mathfrak{B}$  denote the corresponding  $[G \rightarrow \text{Aut}(G)]$ -bundle. The extension is central if, and only if, the  $[G \rightarrow \text{Aut}(G)]$ -bundle  $\mathfrak{B}$  reduces to a principal  $[Z(G) \rightarrow 1]$ -bundle, i.e. there exists a generalized morphism  $Z\mathfrak{B} : \Gamma \rightarrow [Z(G) \rightarrow 1]$  such that*

$$\begin{array}{ccc} \Gamma & \xrightarrow{\mathfrak{B}} & [G \rightarrow \text{Aut}(G)] \\ & \searrow Z\mathfrak{B} & \uparrow \\ & & [Z(G) \rightarrow *] \end{array}$$

*is commutative up to equivalence.*

*Proof.* Let  $\tilde{\Gamma} \xrightarrow{\phi} \Gamma \rightrightarrows M$  be a central  $G$ -extension. The corresponding 2-group bundle  $\mathfrak{B}$  is the generalized morphism  $[M \rightarrow \Gamma] \leftarrow [M \times G \xrightarrow{i} \tilde{\Gamma}] \rightarrow [G \xrightarrow{\text{Ad}} \text{Aut}(G)]$ , see Proposition 3.8 and Remark 3.7. Let  $\tau : \tilde{\Gamma}' \rightarrow \tilde{\Gamma}$  be the inclusion map. The  $Z(G)$ -extension defines the crossed module  $[M \times Z(G) \xrightarrow{i'} \tilde{\Gamma}']$  and we have a commutative diagram

$$\begin{array}{ccccc} [M \rightarrow \Gamma] & \xleftarrow{\quad} & [M \times G \xrightarrow{i} \tilde{\Gamma}] & \xrightarrow{\quad} & [G \xrightarrow{\text{Ad}} \text{Aut}(G)] \\ & \swarrow & \uparrow \tau & & \uparrow \\ & & [M \times Z(G) \xrightarrow{i'} \tilde{\Gamma}'] & \xrightarrow{\quad} & [Z(G) \rightarrow 1] \end{array} \quad (15)$$

Note that the right square in (15) is commutative because the extension is central. Diagram (15) implies that the 2-group bundle  $\mathfrak{B}$  reduces.

Reciprocally, assume  $\mathfrak{B}$  reduces. By Proposition 3.10.(b), passing to a Morita equivalent groupoid, we can assume that the  $G$ -extension is the extension corresponding to the generalized morphism  $\Gamma \xrightarrow{Z\mathfrak{B}} [Z(G) \rightarrow 1] \hookrightarrow [G \rightarrow \text{Aut}(G)]$ . If  $Z\mathfrak{B}$  is the generalized morphism  $[M \rightarrow \Gamma] \leftarrow [M \times L \rightarrow \Delta] \rightarrow [Z(G) \rightarrow 1]$ , it follows from Section 3.2, that  $\tilde{\Gamma} \rightarrow \Gamma \rightrightarrows M$  is the extension  $\tilde{\Gamma} \rightarrow \Gamma \rightrightarrows M$ , where  $\tilde{\Gamma} = (\Delta \times_L Z(G)) \times_{Z(G)} G$ . Since the map  $\Delta \rightarrow \text{Aut}(G)$  is trivial,  $\text{Ad}_{\tilde{\gamma}}$  is trivial for all  $\tilde{\gamma} \in \tilde{\Gamma}$ . Therefore, the extension is central.  $\square$

**Remark 4.10.** *If  $G$  is a reductive Lie group, its Lie algebra  $\mathfrak{g}$  decomposes as a direct sum  $\mathfrak{g} \cong Z(\mathfrak{g}) \oplus \mathfrak{m}$  of ideals, where  $Z(\mathfrak{g})$  is the center of  $\mathfrak{g}$ . In the sequel, the symbol  $\text{pr}$  will denote the induced projection  $\mathfrak{g} \rightarrow Z(\mathfrak{g})$ , which is a homomorphism of Lie algebras and maps  $[\mathfrak{g}, \mathfrak{g}]$  onto 0. Moreover, if  $G$  is connected, this direct sum decomposition is not only  $\text{ad}_{Z(\mathfrak{g})}$ -invariant but also  $\text{Ad}_G$ -invariant and, consequently,  $\text{pr} \circ \text{Ad}_g = \text{pr}$  for all  $g \in G$ . Moreover, for any  $g \in G$  and any smooth path  $t \mapsto f_t$  in  $G$  with  $f_0 = 1$  and  $\frac{d}{dt} f_t|_0 = \xi \in \mathfrak{g}$ , one has*

$$\text{pr} \left( \frac{d}{dt} f_t^{-1} g f_t g^{-1} \Big|_0 \right) = \text{pr}(\text{Ad}_g \xi - \xi) = \text{pr}(\xi) - \text{pr}(\xi) = 0 \quad (16)$$

**Proposition 4.11.** *Let  $\tilde{\Gamma} \xrightarrow{\phi} \Gamma \rightrightarrows M$  be a central  $G$ -extension with  $G$  connected and reductive. Let  $\alpha \in \Omega^1(\tilde{\Gamma}, \mathfrak{g})$  be a connection 1-form for the right principal  $G$ -bundle  $\tilde{\Gamma} \xrightarrow{\phi} \Gamma$ .*

- (a) *Then there exists  $\Omega_\alpha \in Z_{DR}^3(\Gamma_\bullet, Z(\mathfrak{g}))$  such that  $\text{pr}(d\alpha + \partial\alpha) = \phi^*(\Omega_\alpha)$ .*
- (b) *Moreover, if  $\alpha_1$  and  $\alpha_2$  are two different connection 1-forms as above, then  $\Omega_{\alpha_1} - \Omega_{\alpha_2} \in B_{DR}^3(\Gamma_\bullet, Z(\mathfrak{g}))$ .*

We call  $\mathbf{DD}_{(\alpha)} := [\Omega_\alpha] \in H^3(\Gamma) \otimes Z(\mathfrak{g})$  the Dixmier-Douady class of the  $G$ -central extension.

*Proof.* (a) Being a connection 1-form,  $\alpha \in \Omega^1(\tilde{\Gamma}, \mathfrak{g})$  enjoys the following two properties:

$$\begin{aligned} R_g^* \alpha &= \text{Ad}_{g^{-1}} \circ \alpha, \quad \forall g \in G \\ \alpha(\hat{\xi}_x) &= \xi, \quad \forall x \in M, \forall \xi \in \mathfrak{g} \end{aligned}$$

Given any  $\xi \in \mathfrak{g}$  and any  $G$ -invariant vector field  $v \in \mathfrak{X}(\tilde{\Gamma})$ , we get

$$\begin{aligned} d\alpha(\hat{\xi}, v) &= \hat{\xi}(\alpha(v)) - v(\alpha(\hat{\xi})) - \alpha([\hat{\xi}, v]) = \hat{\xi}(\alpha(v)) - v(\xi) - \alpha(\mathcal{L}_{\hat{\xi}} v) \\ &= \mathcal{L}_{\hat{\xi}}(\alpha(v)) = -\text{ad}_{\xi}(\alpha(v)) \end{aligned}$$

since the vector field  $v$  is  $G$ -invariant and the function  $\alpha(v)$  is  $G$ -equivariant. It follows that  $\text{pr} \circ d\alpha(\hat{\xi}, v) = \text{pr}[\alpha(v), \xi] = 0$  since  $\text{pr}[\mathfrak{g}, \mathfrak{g}] = 0$ . Moreover, we have

$$R_g^*(d\alpha) = d(R_g^* \alpha) = d(\text{Ad}_{g^{-1}} \circ \alpha) = \text{Ad}_{g^{-1}} \circ d\alpha$$

for all  $g \in G$ . Therefore, by Remark 4.10, the 2-form  $\text{pr} \circ d\alpha \in \Omega^2(\tilde{\Gamma}, Z(\mathfrak{g}))$  is basic; there exists  $\omega \in \Omega^2(\Gamma, Z(\mathfrak{g}))$  such that  $\text{pr} \circ d\alpha = \phi^* \omega$ .

Consider

$$\tilde{\Gamma}_2 = \tilde{\Gamma} \times_{s, \Gamma, t} \tilde{\Gamma} = \left\{ (x, y) \in \tilde{\Gamma} \times \tilde{\Gamma} \mid s(x) = t(y) \right\},$$

the three face maps

$$p_1(x, y) = x \quad m(x, y) = x \cdot y \quad p_2(x, y) = y$$

from  $\tilde{\Gamma}_2$  to  $\tilde{\Gamma}$  and the action of  $G \times G$  on  $\tilde{\Gamma}_2$  given by

$$(x, y)^{(g, h)} = (xg_{s(x)}, yh_{s(y)}).$$

Then we have

$$\text{pr} \circ d\alpha = \partial(\text{pr} \circ \alpha) = p_2^*(\text{pr} \circ \alpha) - m^*(\text{pr} \circ \alpha) + p_1^*(\text{pr} \circ \alpha).$$

From  $\text{pr} \circ \text{Ad}_g = \text{pr}$  and  $R_g^* \alpha = \text{Ad}_{g^{-1}} \circ \alpha$ , it follows that  $R_g^*(\text{pr} \circ \alpha) = \text{pr} \circ \alpha$ . This, together with the relations  $p_2 \circ R_{(g, h)} = R_h \circ p_2$  and  $p_1 \circ R_{(g, h)} = R_g \circ p_1$  implies that  $p_2^*(\text{pr} \circ \alpha)$  and  $p_1^*(\text{pr} \circ \alpha)$  are invariant under the  $G \times G$ -action  $\tilde{\Gamma}_2$ . Given a smooth path  $t \mapsto (x_t, y_t)$  in  $\tilde{\Gamma}_2$ , one also gets

$$\begin{aligned} & R_{(g, h)}^* m^*(\text{pr} \circ \alpha) \left( \frac{d}{dt} (x_t, y_t) \Big|_0 \right) \\ &= (\text{pr} \circ \alpha) \left( \frac{d}{dt} x_t g y_t h \Big|_0 \right) \\ &= (\text{pr} \circ \alpha) \left( \frac{d}{dt} x_t y_t g^{r(y_t)} h \Big|_0 \right) \\ &= (\text{pr} \circ \alpha) \left( \frac{d}{dt} x_t y_t g^{r(y_0)} h \Big|_0 \right) + (\text{pr} \circ \alpha) \left( \frac{d}{dt} x_0 y_0 g^{r(y_t)} h \Big|_0 \right) \end{aligned}$$

While the first term of the R.H.S. is equal to  $m^*(\text{pr} \circ \alpha) \left( \frac{d}{dt} (x_t, y_t) \Big|_0 \right)$  since  $(\text{pr} \circ \alpha)$  is  $G$ -invariant, the second term vanishes. Indeed, using  $\alpha(\hat{\xi}) = \xi$ ,  $R_h^* \alpha = \text{Ad}_{h^{-1}} \circ \alpha$  and  $\text{pr} \circ \text{Ad}_g = \text{pr}$ , we obtain that

$$\begin{aligned} (\text{pr} \circ \alpha) \left( \frac{d}{dt} x_0 y_0 g^{r(y_t)} h \Big|_0 \right) &= \text{pr} \left( \frac{d}{dt} g^{r(y_t)} (g^{r(y_0)})^{-1} \Big|_0 \right) \\ &= \text{pr} \left( \frac{d}{dt} (g^{r(y_0)})^{r(y_0)^{-1} r(y_t)} (g^{r(y_0)})^{-1} \Big|_0 \right) \end{aligned}$$

and the claim follows from (16). Hence  $R_{(g,h)}^* m^*(\text{pr} \circ \alpha) = m^*(\text{pr} \circ \alpha)$ . Therefore,  $\text{pr} \circ \partial \alpha$  is  $(G \times G)$ -invariant.

One also has

$$\begin{aligned} \text{pr} \circ \partial \alpha \left( \frac{d}{dt} (x e^{t\xi}, y e^{t\eta}) \Big|_0 \right) &= \text{pr} \left( \alpha \left( \frac{d}{dt} y e^{t\eta} \Big|_0 \right) - \alpha \left( \frac{d}{dt} x e^{t\xi} y e^{t\eta} \Big|_0 \right) + \alpha \left( \frac{d}{dt} x e^{t\xi} \Big|_0 \right) \right) \\ &= \text{pr} \left( \eta - \alpha \left( \frac{d}{dt} x y e^{t \text{Ad}_{r(y)}^{-1} \xi} e^{t\eta} \Big|_0 \right) + \xi \right) \\ &= \text{pr}(\eta - \text{Ad}_{r(y)}^{-1} \xi - \eta + \xi) \\ &= \text{pr}(\xi) - \text{pr}(\text{Ad}_{r(y)}^{-1} \xi) \\ &= 0. \end{aligned}$$

Hence the 1-form  $\text{pr} \circ \partial \alpha \in \Omega^1(\tilde{\Gamma}_2, Z(\mathfrak{g}))$  is basic with respect to the principal  $(G \times G)$ -bundle  $\tilde{\Gamma}_2 \rightarrow \Gamma_2$ .

(b) Clearly, one has  $i_\xi(\alpha_1 - \alpha_2) = 0$  and  $R_g^*(\text{pr} \circ (\alpha_1 - \alpha_2)) = \text{pr} \circ (\alpha_1 - \alpha_2)$ . Thus  $\text{pr} \circ (\alpha_1 - \alpha_2) = \phi^* A$ , where  $A \in \Omega^1(\Gamma, Z(\mathfrak{g}))$ . It follows that

$$\begin{aligned} \phi^*(\Omega_{\alpha_1} - \Omega_{\alpha_2}) &= \text{pr}(d(\alpha_1 - \alpha_2) + \partial(\alpha_1 - \alpha_2)) = d(\phi^* A) + \partial(\phi^* A) \\ &= \phi^*(dA + \partial A) \end{aligned}$$

and  $\Omega_{\alpha_1} - \Omega_{\alpha_2} = dA + \partial A \in B^3(\Gamma_\bullet, Z(\mathfrak{g}))$ .  $\square$

**Remark 4.12.** The Dixmier-Douady class  $\mathbf{DD}_{(\alpha)}$  of a central  $G$ -extension identifies with a linear map  $Z(\mathfrak{g})^* \rightarrow H^3(\Gamma)$  by composition with the canonical biduality homomorphism  $Z(\mathfrak{g}) \rightarrow Z(\mathfrak{g})^{**}$ .

**Remark 4.13.** When the group  $G$  is abelian, then  $\tilde{\Gamma}/Z(G) = \Gamma$ ,  $\tilde{\Gamma}' = \tilde{\Gamma}$  and the projection map  $\mathfrak{g} \xrightarrow{\text{pr}} Z(\mathfrak{g}) = \mathfrak{g}$  is the identity. In particular, when  $G = S^1$ , the Dixmier-Douady class given by Proposition 4.11 coincides with the Dixmier-Douady class defined in [7].

## 4.4 Main theorem

Let  $\tilde{\Gamma} \rightarrow \Gamma \rightrightarrows M$  be a  $G$ -central extension of Lie groupoids. According to Proposition 4.9, we obtain a universal characteristic map  $\mathbf{CC}_\Phi : H^3([Z(G) \rightarrow 1]) \rightarrow H^3(\Gamma)$ . According to [20],  $H^3([Z(G) \rightarrow 1])$  is isomorphic to  $Z(\mathfrak{g})^*$  if  $G$  is compact. Thus we obtain a map  $\mathbf{CC}_\Phi : Z(\mathfrak{g})^* \rightarrow H^3(\Gamma)$  which, by duality, defines the *universal characteristic class*  $\mathbf{CC}_\Phi \in H^3(\Gamma) \otimes Z(\mathfrak{g})$ .

The main theorem is

**Theorem 4.14.** *Let  $G$  be a compact connected Lie group. For any  $G$ -central extension of Lie groupoids  $\tilde{\Gamma} \rightarrow \Gamma \rightrightarrows M$ , the universal characteristic class coincides with the Dixmier-Douady class.*

## 4.5 The case of central $S^1$ -extensions

In this Section, as a preliminary step, we prove Theorem 4.14 for the group  $G = S^1$ .

Assume  $\tilde{\Gamma} \xrightarrow{\phi} \Gamma \rightrightarrows M$  is a central  $S^1$ -extension. We consider the following four 2-groupoids:

$$\begin{array}{ll} \mathbf{A} : & \Gamma \rightrightarrows \Gamma \rightrightarrows M \\ \mathbf{B} : & \tilde{\Gamma} \times_\Gamma \tilde{\Gamma} \rightrightarrows \tilde{\Gamma} \rightrightarrows M \\ \mathbf{C} : & Z(G) \rightrightarrows * \rightrightarrows * \\ \mathbf{D} : & \tilde{\Gamma} \rightrightarrows \tilde{\Gamma} \rightrightarrows M \end{array}$$

The central extension  $\phi$  determines the (generalized) morphisms

$$\mathbf{D} \xrightarrow{\phi} \mathbf{A} \quad \text{and} \quad \mathbf{A} \xleftarrow[\sim]{\phi} \mathbf{B} \xrightarrow{f} \mathbf{C}$$

At the nerve level, we get

$$\begin{array}{ccccccc} N_2(\mathbf{D}) & \xrightarrow{\phi} & N_2(\mathbf{A}) & \xleftarrow{\phi} & N_2(\mathbf{B}) & \xrightarrow{f} & N_2(\mathbf{C}) \\ d_0^{\mathbf{D}} \downarrow d_1^{\mathbf{D}} \downarrow d_2^{\mathbf{D}} & & d_0^{\mathbf{A}} \downarrow d_1^{\mathbf{A}} \downarrow d_2^{\mathbf{A}} & & d_0^{\mathbf{B}} \downarrow d_1^{\mathbf{B}} \downarrow d_2^{\mathbf{B}} & & d_0^{\mathbf{C}} \downarrow d_1^{\mathbf{C}} \downarrow d_2^{\mathbf{C}} \\ \tilde{\Gamma} & \xrightarrow{\phi} & \Gamma & \xleftarrow{\phi} & \tilde{\Gamma} & \xrightarrow{f} & * \end{array}$$

where, according to Remark 4.3,

$$\begin{aligned} N_2(\mathbf{A}) &= \Gamma \times_{t, \Gamma_0, s} \Gamma, & N_2(\mathbf{C}) &= Z(G), & N_2(\mathbf{D}) &= \tilde{\Gamma} \times_{t, \Gamma_0, s} \tilde{\Gamma}, \\ N_2(\mathbf{B}) &= \left\{ (a, b, c) \in \tilde{\Gamma}^3 \mid \phi(a) = \phi(b) \text{ and } s(a) = s(b) = t(c) \right\} \\ \phi : N_2(\mathbf{B}) &\rightarrow N_2(\mathbf{A}) : (a, b, c) \mapsto (\phi(a), \phi(c)) \\ f : N_2(\mathbf{B}) &\rightarrow N_2(\mathbf{C}) : (a, b, c) \mapsto ab^{-1} \end{aligned}$$

and the face maps are given by

$$\begin{array}{lll} d_0^{\mathbf{A}}(a, c) = a & d_1^{\mathbf{A}}(a, c) = ac & d_2^{\mathbf{A}}(a, c) = c \\ d_0^{\mathbf{B}}(a, b, c) = a & d_1^{\mathbf{B}}(a, b, c) = bc & d_2^{\mathbf{B}}(a, b, c) = c \\ d_0^{\mathbf{D}}(a, c) = a & d_1^{\mathbf{D}}(a, c) = ac & d_2^{\mathbf{D}}(a, c) = c \end{array}$$

We will need one more map:

$$p_{13} : N_2(\mathbf{B}) \rightarrow N_2(\mathbf{D}) : (a, b, c) \mapsto (a, c)$$

**Lemma 4.15.** *One has*

$$d_0^{\mathbf{B}} = d_0^{\mathbf{D}} \circ p_{13}, \quad d_2^{\mathbf{B}} = d_2^{\mathbf{D}} \circ p_{13}, \quad (17)$$

$$\text{and} \quad d_1^{\mathbf{D}} \circ p_{13}(a, b, c) = f(a, b, c) \cdot d_1^{\mathbf{B}}(a, b, c), \quad \forall (a, b, c) \in N_2(\mathbf{B}). \quad (18)$$

**Lemma 4.16.** *For any pseudo-connection  $\theta \in \Omega(\tilde{\Gamma})$  on the central  $S^1$ -extension  $\tilde{\Gamma} \xrightarrow{\phi} \Gamma \rightrightarrows \Gamma_0$ , one has*

$$\partial^{\mathbf{B}}\theta + f^*(dt) = p_{13}^*(\partial^{\mathbf{D}}\theta)$$

Here  $dt$  denotes the Maurer-Cartan (or angular) form on  $S^1$ .

*Proof.* Since  $\theta(\frac{d}{dt}\tilde{\gamma} \cdot e^{it}|_0) = 1$  and  $\theta$  is  $S^1$ -invariant, it follows from (18) that

$$(d_1^{\mathbf{D}} \circ p_{13})^*\theta = (d_1^{\mathbf{B}})^*\theta + f^*dt \quad (19)$$

Therefore,

$$\begin{aligned} \partial^{\mathbf{B}}\theta - p_{13}^*(\partial^{\mathbf{D}}\theta) &= (d_1^{\mathbf{B}})^*\theta - p_{13}^*(d_1^{\mathbf{D}})^*\theta && \text{by (17),} \\ &= -f^*dt && \text{by (19).} \quad \square \end{aligned}$$

According to Proposition 4.11, the connection  $\theta$  induces a cocycle  $\Omega_\theta \in Z_{DR}^3(\mathbf{A})$ .

**Theorem 4.17.**  $\phi^*[\Omega_\theta] = f^*[dt]$  in  $H^3(\mathbf{B})$

*Proof of Theorem 4.17.* By construction, the cocycle  $\Omega_\theta$  is the sum  $\Omega_\theta = \eta + \omega$ , where  $\phi^*(\eta) = \partial^{\mathbf{D}}\theta$  and  $\phi^*(\omega) = d_{\text{DR}}\theta$ .

$$\begin{aligned} \partial^{\mathbf{B}}\theta + d\theta &= p_{13}^*(\partial^{\mathbf{D}}\theta) - f^*(dt) + d\theta && \text{by Lemma 4.16} \\ &= p_{13}^*(\phi^*\eta) - f^*(dt) + \phi^*\omega \\ &= \phi^*(\eta + \omega) - f^*(dt) \end{aligned} \quad \square$$

Now, Theorem 4.14 in the case  $G = S^1$  follows from Theorem 4.17 since  $\mathbf{CC}_\phi = (\phi^*)^{-1}(f^*(dt))$  and  $\mathbf{DD}_{(\phi)} = [\Omega_\theta]$  (by Proposition 4.11).

## 4.6 Proof of Theorem 4.14

By [26] (see also Section 4.3), the central  $G$ -extension of Lie groupoids  $\tilde{\Gamma} \rightarrow \Gamma \rightrightarrows M$  induces a central  $Z(G)$ -extension  $\tilde{\Gamma}' \rightarrow \Gamma \rightrightarrows M$ , where  $\tilde{\Gamma}' = q^{-1}(\sigma(\Gamma))$ . We recover  $\tilde{\Gamma}$  from  $\tilde{\Gamma}'$  by the formula  $\tilde{\Gamma} \cong (\tilde{\Gamma}' \times G)_{/Z(G)}$ , where the  $Z(G)$ -action on  $\tilde{\Gamma}' \times G$  is given by  $(x, g) \cdot z = (x \cdot z^{-1}, z \cdot g)$  for  $x \in \tilde{\Gamma}'$ ,  $g \in G$  and  $z \in Z(G)$ . The natural inclusion  $\tau : \tilde{\Gamma}' \rightarrow \tilde{\Gamma}$  coincides with the map  $x \mapsto [x, 1_G] \in (\tilde{\Gamma}' \times G)_{/Z(G)}$ . Since  $G$  is compact,  $Z(\mathfrak{g})$  is reductive and we have the Lie algebra morphism  $\text{pr} : \mathfrak{g} \rightarrow Z(\mathfrak{g})$ .

**Lemma 4.18.** *Let  $\alpha \in \Omega^1(\tilde{\Gamma}, \mathfrak{g})$  be a connection 1-form on the right principal  $G$ -bundle  $\tilde{\Gamma} \rightarrow \Gamma$ . Then  $\alpha' := \text{pr}(\tau^*(\alpha)) \in \Omega^1(\tilde{\Gamma}', Z(\mathfrak{g}))$  is a connection 1-form for the right principal  $Z(G)$ -bundle  $\tilde{\Gamma}' \rightarrow \Gamma$ .*

*Proof.* Since the inclusion  $\tau : \tilde{\Gamma}' \hookrightarrow \tilde{\Gamma}$  is  $Z(G)$ -equivariant, we have

$$\alpha'(\hat{\eta}_x) = \text{pr} \circ \alpha \circ \tau_*(\hat{\eta}_x) = \text{pr} \circ \alpha \circ (\hat{\eta}_{\tau(x)}) = \text{pr}(\eta) = \eta$$

for all  $\eta \in Z(\mathfrak{g})$  and  $x \in \tilde{\Gamma}'$ . Similarly, for any  $h \in Z(G)$ , we have

$$R_h^*(\alpha') = R_h^*(\text{pr}(\tau^*(\alpha))) = \text{pr} R_h^* \tau^*(\alpha) = \text{pr} \tau^* R_h^* \alpha = \text{pr} \tau^* \text{Ad}_{h^{-1}} \alpha = \alpha'. \quad \square$$

Since  $\tilde{\Gamma}' \rightarrow \Gamma \rightrightarrows M$  is a  $Z(G)$ -central extension, by Lemma 4.18 and Proposition 4.11, we have the Dixmier-Douady class  $\mathbf{DD}_{(\alpha')} \in H^3(\Gamma) \otimes Z(\mathfrak{g})$ .

**Proposition 4.19.** *One has  $\mathbf{DD}_{(\alpha')} = \mathbf{DD}_{(\alpha)}$ .*

*Proof.* By Lemma 4.18 and Proposition 4.11.(b), we can use the 1-form  $\alpha'$  to calculate the Dixmier-Douady class of  $\tilde{\Gamma}' \rightarrow \Gamma \rightrightarrows M$ . By construction we have a commutative diagram of groupoids morphisms

$$\begin{array}{ccc} \tilde{\Gamma}' & & \\ \tau \downarrow & \searrow \phi' & \\ \tilde{\Gamma} & \xrightarrow{\phi} & \Gamma. \end{array}$$

According to Proposition 4.11.(a), the Dixmier-Douady class  $\mathbf{DD}_{(\alpha)}$  is the cohomology class of the cocycle  $[\Omega_\alpha]$  defined by the identity

$$\text{pr}(d\alpha + \partial\alpha) = \phi^*(\Omega_\alpha). \quad (20)$$



Composing (20) by  $\tau^*$ , we get

$$\begin{aligned}\tau^* \operatorname{pr} (d\alpha + \partial\alpha) &= \tau^* \phi^*(\Omega_\alpha) \\ \operatorname{pr} (d + \partial)\tau^* \alpha &= \phi'^*(\Omega_\alpha) \\ (d + \partial) \operatorname{pr} \tau^* \alpha &= \phi'^*(\Omega_\alpha).\end{aligned}$$

Therefore, by Proposition 4.11,  $\mathbf{DD}_{(\alpha')} = [\Omega_\alpha] = \mathbf{DD}_{(\alpha)}$ .  $\square$

Since  $G$  is compact its center is the quotient  $(Z_0(G) \times C)/N$ , where  $Z_0(G)$  is the connected component of  $1_G$  in  $Z(G)$  and  $C, N$  are finite. We fix an isomorphism of Lie groups  $Z_0(G) \cong S^1 \times \cdots \times S^1$  (with  $n$ -factors). We thus obtain isomorphisms  $Z(\mathfrak{g}) \cong \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_n$  and  $H^3([Z(G) \rightarrow 1]) \cong \mathbb{R}dt_1 \oplus \cdots \oplus \mathbb{R}dt_n$ . Let  $\operatorname{pr}_i : Z(\mathfrak{g}) \rightarrow \mathbb{R}e_i$  ( $i = 1 \dots n$ ) be the natural projection.

**Lemma 4.20.** *One has  $\operatorname{pr}_i(\mathbf{CC}_\Phi) = \mathbf{CC}_\Phi(dt_i)$  in  $H^3(\Gamma)$ .*

*Proof.* Let  $(\xi_1, \dots, \xi_n)$  be the dual basis of  $(e_1, \dots, e_n)$  in  $Z(\mathfrak{g})^*$ . According to [20], the generator  $dt_i$  is the left invariant vector field  $\xi_i^L \in \Omega^1(Z(G)) \subset \Omega^3([Z(G) \rightarrow 1])$  associated to  $\xi_i$ . The Lemma follows.  $\square$

**Proposition 4.21.** *One has  $\mathbf{CC}_\Phi = \mathbf{DD}_{(\alpha')}$*

*Proof.* By linearity and Lemma 4.18, it is sufficient to prove that for all  $i = 1 \dots n$ , one has

$$\operatorname{pr}_i(\mathbf{DD}_{(\theta)}) = \operatorname{pr}_i(\mathbf{CC}_\Phi) = \mathbf{CC}_\Phi(dt_i) \quad (\text{by Lemma 4.20}). \quad (21)$$

The proof of Eq. 21 is similar to that of Theorem 4.17.  $\square$

*Proof of Theorem 4.14.* By Proposition 4.19 and Proposition 4.21 we obtain

$$\mathbf{CC}_\Phi = \mathbf{DD}_{(\alpha')} = \mathbf{DD}_{(\alpha)}$$

and Theorem 4.14 follows.  $\square$

## References

- [1] Paolo Aschieri, Luigi Cantini, and Branislav Jurčo, *Nonabelian bundle gerbes, their differential geometry and gauge theory*, Comm. Math. Phys. **254** (2005), no. 2, 367–400. MR2117631 (2005k:53022)
- [2] John C. Baez, Alissa S. Crans, Urs Schreiber, and Danny Stevenson, *From Loop Groups to 2-Groups*. [arXiv:math/0504123](#).
- [3] John C. Baez and Aaron Lauda, *Higher-dimensional algebra (V). 2-groups*, Theory and Applications of Categories **12** (2004), 423–491. MR2068521 (2005m:18005)
- [4] John C. Baez and Urs Schreiber, *Higher Gauge Theory*, Contemp. Math., vol. 431, Amer. Math. Soc., Providence, RI, 2007. MR2342821
- [5] John C. Baez and Daniel Stevenson, *A classifying space for nonabelian cohomology*. preprint.
- [6] Toby Bartels, *Higher Gauge Theory I: 2-Bundles*. [arXiv:math/0410328](#).
- [7] Kai Behrend and Ping Xu, *Differentiable Stacks and Gerbes*. [arXiv:math/0605694](#).
- [8] Jean Bénabou, *Introduction to bicategories*, Reports of the Midwest Category Seminar, Springer, Berlin, 1967, pp. 1–77. MR0220789 (36 #3841)
- [9] Lawrence Breen, *Bitorseurs et cohomologie non abélienne*, The Grothendieck Festschrift, Vol. I, Progr. Math., vol. 86, Birkhäuser Boston, Boston, MA, 1990, pp. 401–476 (French). MR1086889 (92m:18019)

- [10] Lawrence Breen, *Tannakian categories*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 337–376. MR1265536 (95b:18009)
- [11] Lawrence Breen, *On the classification of 2-gerbes and 2-stacks*, Astérisque **225** (1994), 160 (English, with English and French summaries). MR1301844 (95m:18006)
- [12] Ronald Brown and Kirill C. H. Mackenzie, *Determination of a double Lie groupoid by its core diagram*, J. Pure Appl. Algebra **80** (1992), no. 3, 237–272. MR1170713 (93g:55022)
- [13] M. Bullejos, E. Faro, and V. Blanco, *A full and faithful nerve for 2-categories*, Appl. Categ. Structures **13** (2005), no. 3, 223–233. MR2167791 (2006e:18009)
- [14] Shiing-Shen Chern and Yi-Fone Sun, *The imbedding theorem for fibre bundles*, Trans. Amer. Math. Soc. **67** (1949), 286–303. MR0032996 (11,378c)
- [15] Paul Dedecker, *Sur la cohomologie non abélienne. I*, Canad. J. Math. **12** (1960), 231–251 (French). MR0111021 (22 #1888)
- [16] Paul Dedecker, *Sur la cohomologie non abélienne. II*, Canad. J. Math. **15** (1963), 84–93 (French). MR0143218 (26 #778)
- [17] Johan L. Dupont, *Curvature and characteristic classes*, Springer-Verlag, Berlin, 1978. Lecture Notes in Mathematics, Vol. 640. MR0500997 (58 #18477)
- [18] John W. Duskin, *Simplicial matrices and the nerves of weak  $n$ -categories. I. Nerves of bicategories*, Theory Appl. Categ. **9** (2001/02), 198–308 (electronic). CT2000 Conference (Como). MR1897816 (2003f:18005)
- [19] Ezra Getzler, *Lie Theory for nilpotent  $L_\infty$ -algebras*. [arXiv:0404003](#), to appear in Ann. of Math.
- [20] Grégory Ginot and Ping Xu, *Cohomology of Lie 2-groups*. [arXiv:0712.2069](#).
- [21] Jean Giraud, *Cohomologie non abélienne*, Springer-Verlag, Berlin, 1971 (French). Die Grundlehren der mathematischen Wissenschaften, Band 179. MR0344253 (49 #8992)
- [22] André Haefliger, *Private Communication*.
- [23] André Henriques, *Integrating  $L_\infty$ -algebras*. [arXiv:0603563](#).
- [24] Michel Hilsum and Georges Skandalis, *Morphismes  $K$ -orientés d'espaces de feuilles et fonctorialité en théorie de Kasparov (d'après une conjecture d'A. Connes)*, Ann. Sci. École Norm. Sup. (4) **20** (1987), no. 3, 325–390 (French, with English summary). MR925720 (90a:58169)
- [25] Nigel Hitchin, *Lectures on special Lagrangian submanifolds*, Submanifolds (Cambridge, MA, 1999), AMS/IP Stud. Adv. Math., vol. 23, Amer. Math. Soc., Providence, RI, 2001, pp. 151–182. MR1876068 (2003f:53086)
- [26] Camille Laurent-Gengoux, Mathieu Stiénon, and Ping Xu, *Non Abelian Differential Gerbes*. [arXiv:math/0511696](#).
- [27] Camille Laurent-Gengoux and Friedrich Wagemann, *Obstruction classes of crossed modules of Lie algebroids and Lie groupoids linked to existence of principal bundles*. [arXiv:math/0611226v3](#).
- [28] Masaki Kashiwara and Pierre Schapira, *Categories and sheaves*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 332, Springer-Verlag, Berlin, 2006. MR2182076 (2006k:18001)
- [29] Marco Mackay and Roger Picken, *Holonomy and parallel transport for abelian gerbes*, Adv. Math. **170** (2002), no. 2, 287–339. MR1932333 (2004a:53052)
- [30] Tom Leinster, *Higher operads, higher categories*, London Mathematical Society Lecture Note Series, vol. 298, Cambridge University Press, Cambridge, 2004. MR2094071 (2005h:18030)
- [31] Kirill C. H. Mackenzie, *General theory of Lie groupoids and Lie algebroids*, London Mathematical Society Lecture Note Series, vol. 213, Cambridge University Press, Cambridge, 2005. MR2157566 (2006k:58035)
- [32] John W. Milnor and James D. Stasheff, *Characteristic classes*, Annals of Mathematics Studies, vol. 76, Princeton University Press, Princeton, N. J., 1974. MR0440554 (55 #13428)
- [33] I. Moerdijk, *Lie groupoids, gerbes, and non-abelian cohomology*,  $K$ -Theory **28** (2003), no. 3, 207–258. MR2017529 (2005b:58024)
- [34] I. Moerdijk and J. Mrčun, *Introduction to foliations and Lie groupoids*, Cambridge Studies in Advanced Mathematics, vol. 91, Cambridge University Press, Cambridge, 2003. MR2012261 (2005c:58039)

- [35] M. K. Murray, *Bundle gerbes*, J. London Math. Soc. (2) **54** (1996), no. 2, 403–416. MR1405064 (98a:55016)
- [36] Hisham Sati, James D. Stasheff, and Urs Schreiber, *String 2- and Chern-Simons 3-connections and their generalization*. preprint.
- [37] Graeme Segal, *Classifying spaces and spectral sequences*, Inst. Hautes Études Sci. Publ. Math. **34** (1968), 105–112. MR0232393 (38 #718)
- [38] James D. Stasheff, *H-spaces and classifying spaces: foundations and recent developments*, Algebraic topology (Proc. Sympos. Pure Math., Vol. XXII, Univ. Wisconsin, Madison, Wis., 1970), Amer. Math. Soc., Providence, R.I., 1971, pp. 247–272. MR0321079 (47 #9612)
- [39] Norman Steenrod, *The topology of fibre bundles*, Princeton Landmarks in Mathematics, Princeton University Press, 1999. MR1688579 (2000a:55001)
- [40] Ross Street, *The algebra of oriented simplexes*, J. Pure Appl. Algebra **49** (1987), no. 3, 283–335. MR920944 (89a:18019)
- [41] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR1269324 (95f:18001)
- [42] Chenchang Zhu, *n-Groupoids and Stacky Groupoids*. [arXiv:math/0801.2057](https://arxiv.org/abs/math/0801.2057).